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Functional inequalities and the curvature dimension inequality on totally geodesic foliations

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ABSTRACT

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We discover following analytic / geometric properties on Riemannian foliations with bundle-like metric and totally geodesic leaves, or shortly, totally geodesic foliations. Under a certain curvature condition, we obtain

(1) Sobolev-isoperimetric inequalities, global Poincaré inqualities, and a lower bound for Cheeger's isoperimetric constant,

(2) Poincaré inequalities on balls and uniqueness of positive(or $L^p, p \geq 1$) solutions for the subelliptic heat equation,

(3) A lower bound for the first non-zero eigenvalue of sub-Laplacians (Lichnerowicz theorem), and Obata's sphere theorem.

In this context, the curvature condition is a sub-Riemannian analogue of lower bounds for Ricci curvature tensor. Earlier, it is given by Baudoin-Garofalo's curvature dimension inequality, or Baudoin's Weitzenböck formulas for one forms. Our framework includes CR Sasakian manifolds with Tanaka-Webster (or pseudo-Hermitian) Ricci tensor bounds, K-contact manifolds, and Carnot group of step 2.

1. INTRODUCTION

This thesis consists of the author's research works and results in the sub-Riemannian geometry. The context of the sub-Riemannian geometry requires various modifications of known techniques, such as heat flow methods ([8,10]), associated Markov process with Feynman-Kac formula ([6,14,62]), Riemannian submersions ([13,14]), Saloff-Coste/Grigor'yan's equivalence between doubling+Poincaré, two-sided Gaussian bounds and parabolic Harnack inequality ([42]), etc.
The first project began with functional analysis as an application of the generalized curvature dimension inequality, which gives a certain curvature condition. And lately the range of projects reaches Riemannian foliations/submersions and geometric analysis on differential forms.

1.1 Curvature dimension inequalities

In 2009, F.Baudoin and N.Garofalo introduced sub-Riemannian analogue of Ricci lower bound in the form of a functional inequality - the generalized curvature dimension inequality in [10]. Briefly, for $\rho_2 > 0, \kappa \geq 0$, $d = \dim \mathcal{H}$ ($\mathcal{H} \subset T\mathbb{M}$), $\Gamma:$ Bakry - carré du champ (square of the gradient vector),

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f), \forall \nu > 0$$

$$\Leftrightarrow \text{'Ricci'} \geq \rho_1. \text{ denoted by } CD(\rho_1, \rho_2, \kappa, d).$$

This notion was established on sub-Riemannian manifolds with transverse symmetries. And later, it is extended to the space of Riemannian foliations [14]. The category of such manifolds includes CR Sasakian manifolds, K-contact manifolds, Carnot groups of step 2.

At the beginning, the author's work was focused to disclose various inequalities including Sobolev-isoperimetric inequalities and Poincaré inequalities under the curvature condition above.

In [12], (joint work with my advisor F.Baudoin) we proved the improved Sobolev inequality adapting the idea of Ledoux:

Theorem 1.1.1 *If $CD(0, \rho_2, \kappa, d)$ is satisfied, for any $1 \leq q < p < \infty$, $\theta = p/q$, Besov norm $\|f\|^{\alpha}_{\infty,\infty} = \sup_{t>0} t^{-\alpha/2}\|P_t f\|_\infty$,*

$$\|f\|_q \leq C\|\sqrt{\Gamma(f)}\|_p^{\theta}\|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}.$$

The Besov norm in the last term carries an information about the on-diagonal heat kernel bounds, which is not removable - even on Riemannian manifolds. The improved Sobolev inequality implied the relation between the volume growth condition of balls and isoperimetric inequality. For $D =$ the homogeneous dimension in terms of the volume doubling condition, if $CD(0, \rho_2, \kappa, d)$ is true,

$$\begin{aligned} &\mu(B(x,r)) \geq Cr^D, C > 0, \forall x \in \mathbb{M}, r \geq 0, \\ &\Leftrightarrow \mu(E)^{\frac{D-1}{D}} \leq C'P(E), C' > 0, \text{ for every Caccioppoli set } E \subset \mathbb{M}, \end{aligned}$$

where μ is the measure on $\mathbb{M}$, P is the perimeter, and a Caccioppoli set is a measurable set with finite perimeter.

In [42], the author proved that the Buser's Poincaré inequality on geodesic balls of [19] holds in sub-Riemannian setting.

Theorem 1.1.2 *Let $\mathbb{M}$ be a sub-Riemannian manifold satisfying $CD(-K, \rho_2, \kappa, d)$, $K > 0$. For any $x_0 \in \mathbb{M}$, $r \in (0, \infty)$, $\exists C_1, C_2 > 0$ depending only on ρ_2, κ, d*

$$\int_{B(x_0,r)} |f(x) - f_{B(x_0,r)}|^2 d\mu(x) \leq C_1 r^2 e^{C_2 K r^2} \int_{B(x_0,r)} \Gamma(f) d\mu.$$

Through Moser's iteration and Donnelly's arguments, one can obtain following uniqueness results.

Theorem 1.1.3 *If $CD(-K, \rho_2, \kappa, d)$ is satisfied, for $T > 0$, $f \geq 0$, $\exists! u \geq 0, u \in C(\mathbb{M} \times [0,T))$ s.t.*

$$(L - \frac{\partial}{\partial t})u(x,t) = 0,$$
$$u(x,0) = f(x).$$

1.2 Bochner-Weitzenböck formula

Recently, in [14] (joint work with F.Baudoin and J.Wang), we extended the framework for the generalized curvature dimension inequality into Riemannian foliation with totally geodesic leaves. In fact, in Riemannian foliations, we recovered Weitzenböck-Bochner formula for one-forms which was established by F.Baudoin ([6]) earlier for sub-Riemannian manifolds with transverse symmetries. If we set the bundle-like metric with parameter $\varepsilon > 0$,

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0,$$

for $\forall X, Y \in \Gamma^\infty(\mathcal{H}), Z \in \Gamma^\infty(\mathcal{V})$, $J_Z : \mathcal{H}_x \to \mathcal{H}_x$ is defined as follows,

$$g_{\mathcal{H}}(J_Z(X), Y) = g_{\mathcal{V}}(Z, T(X,Y)).$$

Let us define the 'horizontal divergence of the torsion' as follows (in a local frame),

$$\delta_{\mathcal{H}} T(X) = \sum_{j=1}^{n} (\nabla_{X_j} T)(X_j, X).$$

Finally, for $\eta \in \Gamma^\infty(T^*\mathbb{M})$ and $Y \in \Gamma^\infty(T\mathbb{M})$, if we consider skew-symmetric tensor

$$\mathfrak{T}^\varepsilon_V \eta(Y) = \begin{cases} \frac{1}{\varepsilon}\eta(J_Y V), & Y \in \Gamma^\infty(\mathcal{V}) \\ -\eta(T(V,Y)), Y \in \Gamma^\infty(\mathcal{H}) \end{cases}$$

If ∇ is the Bott connection and $\mathbf{J}^2 = \sum_{\ell=1}^m J_{Z_\ell} J_{Z_\ell}$, by the optimizing argument in [6], a sub-Laplacian on one-forms is defined as follows

$$\square_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) - \frac{1}{\varepsilon}\mathbf{J}^2 + \frac{1}{\varepsilon}\delta_{\mathcal{H}} T - \mathfrak{Ric}_{\mathcal{H}}.$$

We recover Baudoin's sub-Riemannian Bochner-Weitzenböck formula for one forms, $\forall f \in C^\infty(\mathbb{M}), \forall \eta \in \Gamma^\infty(T^*\mathbb{M})$

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta\rangle_\varepsilon = \|\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^\varepsilon \eta\|_\varepsilon^2 + \langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta\rangle_{\mathcal{H}} - \langle \delta_{\mathcal{H}} T(\eta), \eta\rangle_{\mathcal{V}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_{\mathcal{H}},$$

$$dLf = \square_\varepsilon df.$$

When ∇ is Yang-Mills type, i.e. $\delta_{\mathcal{H}} T = 0$, $\square_\varepsilon$ is symmetric. Then we managed to prove that $\mathbb{M}$ is stochastically complete and the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied if we assume that $\mathbb{M}$ is complete and that globally on $\mathbb{M}$, for every $\eta_1 \in \Gamma^\infty(\mathcal{H}^*)$ and $\eta_2 \in \Gamma^\infty(\mathcal{V}^*)$,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta_1), \eta_1\rangle_{\mathcal{H}} \geq \rho_1\|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2\eta_1, \eta_1\rangle_{\mathcal{H}} \leq \kappa\|\eta_1\|_{\mathcal{H}}^2, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_{\eta_2}^2) \geq \rho_2\|\eta_2\|_{\mathcal{V}}^2.$$

Therefore, in the context of Riemannian foliation with totally geodesic leaves, we recover all the results in [7–12, 42],i.e., if $\mathbb{M}$ satisfies the above condition, we have

Proposition 1.2.1 *(Li-Yau type inequalities) ([10]) For any bounded $f \in C^\infty(\mathbb{M})$, such that $f, \sqrt{\Gamma(f)}$, $\sqrt{\Gamma^{\mathcal{V}}(f)} \in L_\mu^2(\mathbb{M})$, $f \geq 0$, $f \neq 0$, the following inequality holds for $t > 0$:*

$$\begin{aligned}&\Gamma(\ln P_t f) + \frac{2\rho_2}{3}t\Gamma^{\mathcal{V}}(\ln P_t f)\\ \leq &\left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3}t\right)\frac{LP_t f}{P_t f} + \frac{n\rho_1^2}{6}t - \frac{n\rho_1}{2}\left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{n\left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}.\end{aligned}$$

Proposition 1.2.2 *(Gaussian lower and upper bounds for the horizontal heat kernel) ([8]) If $\rho_1 \geq 0$, then for any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(n, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \to 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has*

$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))}\exp\left(-\frac{Dd(x,y)^2}{n(4-\varepsilon)t}\right) \leq p(x,y,t) \leq \frac{C(\varepsilon)}{\mu(B(x,\sqrt{t}))}\exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right).$$

Here $D = \left(1 + \frac{3\kappa}{2\rho_2}\right)n$ and $d(x,y)$ is the sub-Riemannian distance between x and y.

Proposition 1.2.3 *(Bonnet-Myers theorem) ([10]) Suppose that $\rho_1 > 0$. Then, the manifold $\mathbb{M}$ is compact and the sub-Riemannian diameter of $\mathbb{M}$ satisfies the bound*

$$\textit{diam}\ \mathbb{M} \leq 2\sqrt{3}\pi\sqrt{\frac{\kappa + \rho_2}{\rho_1\rho_2}\left(1 + \frac{3\kappa}{2\rho_2}\right)n}.$$

In particular, (joint work with F.Baudoin) in [13], we obtained the sharp eigenvalue bound for the sub-Laplacian.

Theorem 1.2.1 (Lichnerowicz-Obata theorem) *Assume $\rho_1 > 0$. Then the first eigenvalue λ_1 of the horizontal Laplacian $-L$ satisfies*

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{n} + \frac{3\kappa}{4\rho_2}}.$$

In addition, assume that $\mathbb{M}$ is of H-type. If there is a nontrivial eigenfunction realizing the equality of the eigenvalue bound, then $\mathbb{M}$ is equivalent to a 1-Sasakian sphere $\mathbb{S}^{2m+1}(r)$ or a 3-Sasakian sphere $\mathbb{S}^{4m+3}(r)$ for some $r > 0$ and $m \geq 1$.

This theorem is the extension of known results on CR contact/quaternionic manifolds [36, 38] to a large class of sub-Riemannian manifolds.

2. SOBOLEV AND ISOPERIMETRIC INEQUALITIES

2.1 Introduction and framework

Let $\mathbb{M}$ be a C^∞ connected finite dimensional manifold endowed with a smooth measure μ and a second-order diffusion operator L on $\mathbb{M}$, locally subelliptic in the sense of [27], [41], $L1 = 0$ is satisfied, and

$$\int_{\mathbb{M}} fLg d\mu = \int_{\mathbb{M}} gLf d\mu, \qquad \int_{\mathbb{M}} fLf d\mu \leq 0,$$

for every $f, g \in C_0^\infty(\mathbb{M})$. We indicate with $\Gamma(f) := \Gamma(f,f)$ the *carré du champ* of L, that is the quadratic differential form defined by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad f, g \in C^\infty(\mathbb{M}). \tag{2.1}$$

There is an intrinsic distance associated to L that can be defined via the notion of subunit curves (see [27]). An absolutely continuous curve $\gamma : [0,T] \to \mathbb{M}$ is said to be subunit for the operator L if for every smooth function $f : \mathbb{M} \to \mathbb{R}$ we have $\left|\frac{d}{dt} f(\gamma(t))\right| \leq \sqrt{(\Gamma f)(\gamma(t))}$. We then define the subunit length of γ as $\ell_s(\gamma) = T$. Given $x, y \in \mathbb{M}$, we indicate with

$$S(x,y) = \{\gamma : [0,T] \to \mathbb{M} \mid \gamma \text{ is subunit for } L, \gamma(0) = x, \ \gamma(T) = y\}.$$

In this chapter we assume that $S(x,y)$ is not empty for every $x, y \in \mathbb{M}$. Under such assumption it is easy to verify that

$$d(x,y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x,y)\}, \tag{2.2}$$

defines a true distance on $\mathbb{M}$. Furthermore, in that case, it is known that

$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \leq 1\}, \quad x, y \in \mathbb{M}. \tag{2.3}$$

Throughout this article, we assume that the metric space $(\mathbb{M}, d)$ is complete.

In addition to the differential form (2.1), we assume that $\mathbb{M}$ is endowed with another smooth symmetric bilinear differential form, indicated with Γ^Z, satisfying for $f, g \in C^\infty(\mathbb{M})$

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),$$

and $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$.

We make the following assumptions that will be in force throughout the chapter:

(H.1) There exists an increasing sequence $h_k \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on $\mathbb{M}$, and

$$||\Gamma(h_k)||_\infty + ||\Gamma^Z(h_k)||_\infty \to 0, \quad \text{as } k \to \infty.$$

(H.2) For any $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

(H.3) For every $t \geq 0$, $P_t 1 = 1$ and for every $f \in C_0^\infty(\mathbb{M})$ and $T \geq 0$, one has

$$\sup_{t \in [0,T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty,$$

where P_t is the heat semigroup generated by L.

As it has been proved in [10], the assumption (H.1) implies in particular that L is essentially self-adjoint on $C_0^\infty(\mathbb{M})$. The assumption (H.2) is more subtle and is crucial for the validity of most the subsequent results: It is discussed in details in [10] in several geometric examples. In the sub-Riemannian geometries covered by the present work (H.2) means that the torsion of the sub-Riemannian connection is vertical. Assumption (H.3) is necessary to rigorously justify the Bakry-Émery type arguments. It is a consequence of the generalized curvature dimension inequality below in many examples (see [10]).

In addition to Γ and Γ^Z we need the following second order differential bilinear forms:

$$\Gamma_2(f,g) = \frac{1}{2}\big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)\big], \tag{2.4}$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}\big[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)\big]. \tag{2.5}$$

As for Γ and Γ^Z, we will freely use the notations $\Gamma_2(f) = \Gamma_2(f,f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

The following curvature dimension condition was introduced in [10].

Definition 2.1.1 *(See [10]) We say that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(\rho_1,\rho_2,\kappa,d)$ *if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < d < \infty$ such that the inequality*

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f)$$

holds for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$, where Γ_2 and Γ_2^Z are defined by (2.4) and (2.5).

The motivation for such criterion comes from the study of several examples coming from sub-Riemannian geometry where the generalized curvature dimension inequality turns out to be equivalent to lower bounds on intrinsic curvature tensors (see [10]). The parameter ρ_1 is of special importance, it is the curvature parameter. The condition $\rho_1 = 0$ means that the ambient space has a non negative curvature whereas the condition $\rho_1 > 0$ means that it has a positve curvature. In particular, in the latter case a Bonnet-Myers type theorem was proved in [10], implying that $\mathbb{M}$ needs to be compact.

Our goal in the present work will be to discuss Sobolev type embeddings, isoperimetric type results and Poincaré inequalities by using the generalized curvature dimension inequality. Our methods will exploit and extend to the present subelliptic framework some clever and beautiful ideas due to M. Ledoux ([49]) who used heat semigroup methods to study isoperimetric, Sobolev and Poincaré inequalities. Our discussion will be based on the curvature parameter ρ_1.

In the case $\rho_1 = 0$, which is studied in Section 2, one of our main results is the following Besov-Sobolev embedding:

Theorem 2.1.1 *Assume that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(0, \rho_2, \kappa, d)$. *For every $1 \le p < q < \infty$ and every $f \in W^{1,p}(\mathbb{M})$, we have*

$$\|f\|_q \le C \|\sqrt{\Gamma(f)}\|_p^{\theta} \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}$$

where $\theta = \frac{p}{q}$, where $C > 0$ is a constant that only depends on p, q, ρ_2, κ, d and where $\|\cdot\|_{B^{\theta/(\theta-1)}_{\infty,\infty}}$ is the Besov norm which is introduced in (2.9).

We then prove that this Besov-Sobolev embedding implies the following isoperimetric inequality:

Proposition 2.1.1 *Assume that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(0, \rho_2, \kappa, d)$. *Assume that there exists constants $C > 0$ and $D > 0$ such that for every $x \in \mathbb{M}$, $R \ge 0$, $\mu(B(x, R)) \ge CR^D$. For any $1 \le p, q, r < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$, there exists a constant $C' > 0$ such that $\forall f \in C_0^\infty(\mathbb{M})$, we have*

$$\|f\|_q \le C' \|\sqrt{\Gamma(f)}\|_p^{p/q} \quad \|f\|_r^{1-p/q},$$

and there exists a constant $C'' > 0$ such that for every Caccioppoli set $E \subset \mathbb{M}$ one has

$$\mu(E)^{\frac{D-1}{D}} \le C'' P(E), \tag{2.6}$$

where $P(E)$ denotes the horizontal perimeter of E in $\mathbb{M}$.

In the isoperimetric inequality (2.6) the constant C'' we obtain is not sharp but the exponent $\frac{D-1}{D}$ is correct as the example of the Heisenberg group, to which the result applies, shows. We can observe that in the Euclidean case the optimal isoperimetric constant can be obtained from the semigroup method by using Riesz-Sobolev rearrangement type inequalities. But, so far, to the knowledge of the authors, the rearrangement inequality is not available in the Heisenberg group case. Since the celebrated paper of Pansu [57], the problem of the optimal isoperimetric constant on the Heisenberg group is a long-standing open problem.

In the Section 3, we study the case where the curvature parameter ρ_1 is positive. In that case, as we stressed it before, the manifold $\mathbb{M}$ needs to be compact and the measure μ finite. We obtain the following Poincaré inequality:

Proposition 2.1.2 *Assume that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(\rho_1, \rho_2, \kappa, d)$ *with $\rho_1 > 0$. Let $1 \le p < \infty$. There exists $C = C_p(\rho_1, \rho_2, \kappa, d) > 0$ such that for every $f \in C_0^\infty(\mathbb{M})$,*

$$\|f - f_{\mathbb{M}}\|_p \le C \|\sqrt{\Gamma(f)}\|_p,$$

where $f_{\mathbb{M}} = \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu$.

Interestingly, the constant C we obtain is explicit enough and does not depend on p in for $1 \le p < 2$ or $2 \le p < \infty$. Also C does not depend on the dimension d when $1 \le p < 2$.

The end of Section 3 is then devoted to the study of the isoperimetric constant introduced by Cheeger in [20] in a Riemannian framework and to the study of the first non zero eigenvalue of $\mathbb{M}$. Concerning the Cheeger's isoperimetric constant, we prove in particular the following lower bound:

Proposition 2.1.3 *Assume that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(\rho_1, \rho_2, \kappa, d)$ *with $\rho_1 > 0$ and that $\mu(\mathbb{M}) = 1$. Define*

$$\iota = \inf \frac{P(E)}{\mu(E)},$$

where the infimum runs over all Caccioppoli sets E such that $\mu(E) \le \frac{1}{2}$. We have then

$$\iota \ge \frac{1}{2}\sqrt{\frac{\rho_1}{2}}\frac{1}{1+\frac{2\kappa}{\rho_2}}.$$

And concerning the first eigenvalue we prove the following analogue of the celebrated Lichnerowicz' lower bound:

Proposition 2.1.4 *Assume that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(\rho_1, \rho_2, \kappa, d)$ *with $\rho_1 > 0$. The first non zero eigenvalue λ_1 of $-L$ satisfies the estimate*

$$\lambda_1 \geq \frac{\rho_1 \rho_2}{\frac{d-1}{d}\rho_2 + \kappa}.$$

To conclude this introduction, let us now turn to the fundamental question of examples to which the above results apply. We refer the reader to [10] for more details about most of the examples we discuss below.

Besides Laplace-Beltrami operators on complete Riemannian manifolds with Ricci curvature bounded from below, a wide class of examples is given by sub-Laplacians on sub-Riemannian manifold with transverse symmetries. Sub-Laplacians on Sasakian manifolds form a special and interesting subclasses that we quickly describe below. Let $\mathbb{M}$ be a complete strictly pseudo convex CR Sasakian manifold with real dimension $2n+1$. Let θ be a pseudo-hermitian form on $\mathbb{M}$ with respect to which the Levi form is positive definite. The kernel of θ determines an horizontal bundle $\mathcal{H}$. Denote now by T the Reeb vector field on $\mathbb{M}$, i.e., $d\theta(T, \cdot) = 0, \theta(T) = 1$. We recall that the CR manifold $(\mathbb{M}, \theta)$ is called Sasakian if T is a sub-Riemannian Killing field. For instance the standard CR structures on the Heisenberg group $\mathbb{H}_{2n+1}$ and the sphere $\mathbb{S}^{2n+1}$ are Sasakian. On CR manifolds, there is a canonical subelliptic diffusion operator which is called the CR sub-Laplacian. It is analogous to the Laplace-Beltrami operator in Riemannian geometry. In this framework we have the following result that shows the relevance of the generalized curvature dimension inequality.

Proposition 2.1.5 *[10] Let $(\mathbb{M}, \theta)$ be a complete* CR *Sasakian manifold with real dimension $2n+1$. If for every $x \in \mathbb{M}$ the Tanaka-Webster Ricci tensor satisfies the bound*

$$\mathrm{Ric}_x(v, v) \;\geq \rho_1 |v|^2,$$

for every horizontal vector $v \in \mathcal{H}_x$, then, for the CR sub-Laplacian of $\mathbb{M}$*, the curvature dimension inequality* $\mathrm{CD}(\rho_1, \frac{d}{4}, 1, d)$ *holds with $d = 2n$ and $\Gamma^Z(f) = (Tf)^2$ and the hypothesis (H.1),(H.2),(H.3) are satisfied.*

In addition to sub-Laplacians on Heisenberg groups, more generally, the sub-Laplacian on any Carnot group of step 2 has been shown to satisfy the generalized curvature dimension inequality $CD(0, \rho_2, \kappa, d)$, for some values of the parameters ρ_2 and κ.

2.2 The case $\rho_1 = 0$

Throughout the Section 2, we assume that L satisfies the *generalized curvature dimension inequality* $CD(0, \rho_2, \kappa, d)$ with $\rho_2 > 0$ and $\kappa \geq 0$.

The main tool to prove the theorems mentioned in the introduction, is the heat semigroup $P_t = e^{tL}$, which is defined using the spectral theorem. Since L satisfies the curvature dimension inequality, this semigroup is stochastically complete (see [10]), i.e. $P_t 1 = 1$. Moreover, thanks to the hypoellipticity of L, for $f \in L^p(\mathbb{M})$, $1 \leq p \leq \infty$, the function $(t, x) \to P_t f(x)$ is smooth on $\mathbb{M} \times (0, \infty)$ and

$$P_t f(x) = \int_{\mathbb{M}} p(x, y, t) f(y) d\mu(y)$$

where $p(x, y, t) = p(y, x, t) > 0$ is the so-called heat kernel associated to P_t.

A key ingredient in the following analysis is the following gradient bound that was proved in [10].

Theorem 2.2.1 (Li-Yau type gradient estimate with $\rho_1 = 0$) *Let $f \in C_0^\infty(\mathbb{M})$, $f \geq 0$, $f \not\equiv 0$, then the following inequality holds for $t > 0$:*

$$\Gamma(\ln P_t f) \leq \left(1 + \frac{3\kappa}{2\rho_2}\right) \frac{LP_t f}{P_t f} + \frac{d\left(1 + \frac{3\kappa}{2\rho_2}\right)^2}{2t}.$$

2.2.1 Gradient bounds for the heat semigroup

Proposition 2.2.1 *Let $f \in C_0^\infty(\mathbb{M})$.*

- *If $1 \le p < 2$, then for every $t > 0$,*

$$\left\| \sqrt{\Gamma(P_t f)} \right\|_p \le \frac{1+\frac{3\kappa}{2\rho_2}}{\sqrt{1+(p-1)\left(1+\frac{3\kappa}{2\rho_2}\right)}} \sqrt{\frac{d}{2t}} \|f\|_p.$$

- *If $2 \le p \le +\infty$, then for every $t > 0$,*

$$\left\| \sqrt{\Gamma(P_t f)} \right\|_p \le \sqrt{\frac{1+\frac{2\kappa}{\rho_2}}{2t}} \|f\|_p.$$

Proof Suppose that $1 \le p < 2$.

By Theorem 2.2.1, for $f \in C_0^\infty(\mathbb{M})$, $f \ge 0$, $f \not\equiv 0$, $t > 0$,

$$(P_t f)^{p-2}\Gamma(P_t f) \le \frac{D}{d}(P_t f)^{p-1}(LP_t f) + \frac{D^2}{2td}(P_t f)^p,$$

where $D = d(1+\frac{3\kappa}{2\rho_2})$. It follows that

$$\begin{aligned}
\int_{\mathbb{M}} (P_t f)^{p-2}\Gamma(P_t f) d\mu &\le \frac{D}{d}\int_{\mathbb{M}} (P_t f)^{p-1}(LP_t f) d\mu + \frac{D^2}{2td}\int_{\mathbb{M}} (P_t f)^p d\mu \\
&= -\frac{D}{d}\int_{\mathbb{M}} \Gamma((P_t f)^{p-1}, P_t f) d\mu + \frac{D^2}{2td}\int_{\mathbb{M}} (P_t f)^p d\mu \\
&= -\frac{D}{d}\int_{\mathbb{M}} (p-1)(P_t f)^{p-2}\Gamma(P_t f) d\mu + \frac{D^2}{2td}\int_{\mathbb{M}} (P_t f)^p d\mu.
\end{aligned}$$

Observing $\int_{\mathbb{M}} (P_t f)^p d\mu = \|P_t f\|_p^p \le \|f\|_p^p$, we get

$$\int_{\mathbb{M}} (P_t f)^{p-2}\Gamma(P_t f) d\mu \le \frac{1}{1+(p-1)\frac{D}{d}} \left(\frac{D^2}{2td}\right) \|f\|_p^p.$$

On the other hand, let us pick $\alpha = \frac{p}{2}, \beta = \frac{2-p}{2}$. Since $1 \le p < 2$, one can easily check that

$$\begin{aligned}
\left(\int_{\mathbb{M}} (P_t f)^{p-2}\Gamma(P_t f) d\mu\right)^\alpha &= \left\| (P_t f)^{\frac{p(p-2)}{2}} \Gamma(P_t f)^{\frac{p}{2}} \right\|_{\frac{2}{p}}, \\
\left(\int_{\mathbb{M}} (P_t f)^p d\mu\right)^\beta &= \left\| (P_t f)^{\frac{p(2-p)}{2}} \right\|_{\frac{2}{2-p}}.
\end{aligned}$$

So, by Hölder's inequality, we obtain

$$\int_{\mathbb{M}} \Gamma(P_t f)^{\frac{p}{2}} d\mu \le \left(\int_{\mathbb{M}} (P_t f)^{p-2}\Gamma(P_t f) d\mu\right)^\alpha \left(\int_{\mathbb{M}} (P_t f)^p d\mu\right)^\beta,$$

or equivalently,

$$\begin{aligned}
\int_{\mathbb{M}} (P_t f)^{p-2}\Gamma(P_t f) d\mu &\geq \left[\int_{\mathbb{M}} \Gamma(P_t f)^{\frac{p}{2}} d\mu \left(\int_{\mathbb{M}} (P_t f)^p d\mu \right)^{-\beta} \right]^{\frac{1}{\alpha}} \\
&= \left[\| \sqrt{\Gamma(P_t f)} \|_p^p \; \| P_t f \|_p^{-p\beta} \right]^{\frac{1}{\alpha}} \\
&= \left[\| \sqrt{\Gamma(P_t f)} \|_p^p \; \| P_t f \|_p^{-p(2-p)/2} \right]^{\frac{2}{p}} \\
&= \| \sqrt{\Gamma(P_t f)} \|_p^2 \; \| P_t f \|_p^{-(2-p)}
\end{aligned}$$

Therefore, for $1 \leq p < 2$, we obtain

$$\begin{aligned}
\| \sqrt{\Gamma(P_t f)} \|_p^2 &\leq \left[\frac{1}{1 + (p-1)\frac{D}{d}} \left(\frac{D^2}{2td} \right) \| f \|_p^p \right] \| P_t f \|_p^{2-p} \\
&\leq \frac{1}{1 + (p-1)\frac{D}{d}} \left(\frac{D^2}{2td} \right) \| f \|_p^p \| f \|_p^{2-p} \\
&= \frac{1}{1 + (p-1)\frac{D}{d}} \left(\frac{D^2}{2td} \right) \| f \|_p^2.
\end{aligned}$$

For $f \in C_0^\infty(\mathbb{M})$, let us decompose $f = f^+ - f^-$, where $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$.

Then for each of f^+ and f^-, the above gradient estimate holds.

We can then finish the proof by observing that $\|f\|_p = \|f^+\|_p + \|f^-\|_p$ and $\| \sqrt{\Gamma(P_t f)} \|_p \leq \| \sqrt{\Gamma(P_t f^+)} + \sqrt{\Gamma(P_t f^-)} \|_p \leq \| \sqrt{\Gamma(P_t f^+)} \|_p + \| \sqrt{\Gamma(P_t f^-)} \|_p$.

Now suppose that $2 \leq p \leq +\infty$.

In [7], the following reverse Poincaré inequality (Caccioppoli type inequality) is proved:

$$\Gamma(P_t f) + \rho_2 t \Gamma^Z(P_t f) \leq \frac{1 + \frac{2\kappa}{\rho_2}}{2t} \left(P_t(f^2) - (P_t f)^2 \right).$$

For $2 \leq p \leq +\infty$, one can write $\|P_t(f^2)\|_{\frac{p}{2}} \leq \|f^2\|_{\frac{p}{2}} = \|f\|_p^2$.

Therefore, we have

$$\begin{aligned}
\|\Gamma(P_t f)\|_{\frac{p}{2}} &\leq \frac{1 + \frac{2\kappa}{\rho_2}}{2t} \left\| P_t(f^2) \right\|_{\frac{p}{2}} \\
&\leq \frac{1 + \frac{2\kappa}{\rho_2}}{2t} \|f\|_p^2,
\end{aligned}$$

which implies

$$\left\|\sqrt{\Gamma(P_t f)}\right\|_p \le \sqrt{\frac{1+\frac{2\kappa}{\rho_2}}{2t}}\|f\|_p.$$

■

2.2.2 Pseudo-Poincaré inequalities

By duality, the previous gradient bounds lead to the following pseudo-Poincaré type inequalities:

Proposition 2.2.2 *Let $f \in C_0^\infty(\mathbb{M})$.*

- *If $1 \le p < 2$, then for every $t \ge 0$,*

$$\|f - P_t f\|_p \le \sqrt{\left(2+\frac{4\kappa}{\rho_2}\right) t}\|\sqrt{\Gamma(f)}\|_p \tag{2.7}$$

- *If $2 \le p \le +\infty$, then for every $t \ge 0$,*

$$\|f - P_t f\|_p \le \frac{\left(1+\frac{3\kappa}{2\rho_2}\right)\sqrt{2d}}{\sqrt{1+(p-1)\left(1+\frac{3\kappa}{2\rho_2}\right)}}\sqrt{t}\|\sqrt{\Gamma(f)}\|_p \tag{2.8}$$

Proof Let $p' = \frac{p}{p-1}$. For any $g \in C_0^\infty(\mathbb{M})$ with $\|g\|_{p'} \le 1$, we have

$$\begin{aligned}
\int_{\mathbb{M}} g(f - P_t f) d\mu &= \int_{\mathbb{M}} g\left(-\int_0^t \partial_s P_s f ds\right) d\mu \\
&= -\int_0^t \int_{\mathbb{M}} g L P_s f d\mu ds = -\int_0^t \int_{\mathbb{M}} g P_s L f d\mu ds \\
&= -\int_0^t \int_{\mathbb{M}} P_s g L f d\mu ds = \int_0^t \int_{\mathbb{M}} \Gamma(P_s g, f) d\mu ds \\
&\le \|\sqrt{\Gamma(f)}\|_p \int_0^t \|\sqrt{\Gamma(P_s g)}\|_{p'} ds.
\end{aligned}$$

By using Proposition 2.2.1, we have

$$\int_0^t \|\sqrt{\Gamma(P_s g)}\|_{p'} ds \le \int_0^t \frac{C_{p'}}{\sqrt{s}} ds \|g\|_{p'}.$$

We therefore obtain

$$\int_{\mathbb{M}} g(f - P_t f) d\mu \le 2C_{p'} \sqrt{t} \| \sqrt{\Gamma(f)} \|_p \|g\|_{p'}.$$

By duality we can now conclude that

$$\|f - P_t f\|_p \le 2C_{p'} \sqrt{t} \| \sqrt{\Gamma(f)} \|_p.$$

■

2.2.3 Improved Sobolev embedding

For $\alpha < 0$, we define the Besov norm $\| \cdot \|_{B^{\alpha}_{\infty,\infty}}$ on $\mathbb{M}$ as follows :

$$\|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{t>0} t^{-\alpha/2} \|P_t f\|_\infty. \tag{2.9}$$

It is clear from this definition that $\|f\|_{B^{\alpha}_{\infty,\infty}} \le 1$ is equivalent to the fact that for every $u > 0$, $|P_{t_u} f| \le u$ where $t_u = u^{2/\alpha}$. For $p \ge 1$, we define then the Sobolev space $W^{1,p}(\mathbb{M})$ as the closure of $C_0^\infty(\mathbb{M})$ with respect to the norm $\|f\|_p + \| \sqrt{\Gamma(f)} \|_p$.

Theorem 2.2.2 (Improved Sobolev embedding) *For every $1 \le p < q < \infty$ and every $f \in W^{1,p}(\mathbb{M})$, we have*

$$\|f\|_q \le C \| \sqrt{\Gamma(f)} \|_p^\theta \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}} \tag{2.10}$$

where $\theta = \frac{p}{q}$ and where $C > 0$ is a constant that only depends on p, q, ρ_2, κ, d.

Proof Techniques of the proof are mainly based on [49]; for the sake of completeness, we reproduce the main arguments and make sure they adapt to our sub-Riemannian framework. The proof proceeds in three steps.

Step 1. We first prove the weak-type inequality

$$\|f\|_{q,\infty} \le C \| \sqrt{\Gamma(f)} \|_p^\theta \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}}.$$

Without loss of generality, we can assume $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \le 1$, which is equivalent to the condition:

$$|P_{t_u} f| \le u \ , \ t_u = u^{2(\theta-1)/\theta} \text{ for every } u > 0. \tag{2.11}$$

We have then

$$u^q\mu\{|f|>2u\} \quad\leq\quad u^q\mu\{|f-P_{t_u}f|>u\} \quad\leq\quad u^{q-p}\int_M |f-P_{t_u}f|^p d\mu$$

From Proposition 2.2.2, we have

$$\|f-P_tf\|_p \leq C\sqrt{t}\|\sqrt{\Gamma(f)}\|_p.$$

Since $q-p+\frac{p}{2}\frac{2(\theta-1)}{\theta}=0$, we conclude

$$\begin{aligned} u^q\mu\{|f|>2u\} \leq & u^{q-p}\left(C^p t_u^{p/2}\|\sqrt{\Gamma(f)}\|_p^p\right) \\ \leq & C^p\|\sqrt{\Gamma(f)}\|_p^p \end{aligned}$$

We finally observe that $\sup_{u>0} u^q\mu\{|f|>2u\}=\frac{1}{2^q}\|f\|_{q,\infty}^q$, to conclude Step 1.

Step 2. In the previous weak type inequality, we would like to replace the $L^{q,\infty}$-norm by the L^q-norm. Again, we assume $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}}\leq 1$, that is $|P_{t_u}f|\leq u$ for $t_u=u^{2(\theta-1)/\theta}$, $\forall u>0$. For $f\in W^{1,p}(\mathbb{M})\cap L^q(\mathbb{M})$ such that $|P_{t_u}f|\leq u$, $\forall u>0$, we want to show that for some constant $C>0$,

$$\int_M |f|^q d\mu \leq C\int_{\mathbb{M}} \Gamma(f)^{p/2}d\mu.$$

Let $c\geq 5$ be an arbitrary constant. For any $u>0$, we introduce the truncation

$$\tilde{f}_u=(f-u)^+\wedge((c-1)u)+(f+u)^-\vee(-(c-1)u).$$

That is, $\tilde{f}_u(x)=f(x)-u$ when $u\leq f(x)\leq cu$, and $\tilde{f}_u(x)=f(x)+u$ when $-cu\leq f(x)\leq -u$, otherwise $|\tilde{f}_u|$ is truncated as constants 0 or $(c-1)u$. Observing

$$\{|f|\geq 5u\}\subset\{|\tilde{f}_u|\geq 4u\},$$

yields

$$\begin{aligned} \int_0^\infty \mu(\{|f|\geq 5u\})d(u^q) &\leq \int_0^\infty \mu(\{|\tilde{f}_u|\geq 4u\})d(u^q) \\ &\leq \int_0^\infty \mu(\{|\tilde{f}_u-P_{t_u}f|\geq 3u\})d(u^q) \quad (\text{ since } |P_{t_u}(f)|\leq u) \\ &\leq \int_0^\infty \mu(\{|\tilde{f}_u-P_{t_u}\tilde{f}_u|\geq u\})d(u^q)+\int_0^\infty \mu(\{P_{t_u}(|f-\tilde{f}_u|)\geq 2u\})d(u^q). \end{aligned}$$

We now apply the pseudo-Poincaré inequality for $\tilde{f}_u$ as follows,

$$\begin{aligned}
\mu(\{|\tilde{f}_u - P_{t_u}\tilde{f}_u| \geq u\}) \leq & u^{-p}\int_{\mathbb{M}} |\tilde{f}_u - P_{t_u}\tilde{f}_u|^p d\mu \\
\leq & C' u^{-p} t_u^{p/2} \int_{\mathbb{M}} \Gamma(\tilde{f}_u)^{p/2} d\mu \\
= & C' u^{-q} \int_{\{u \leq |f| \leq cu\}} \Gamma(f)^{p/2} d\mu.
\end{aligned}$$

So by integration we get,

$$\begin{aligned}
\int_0^\infty \mu(\{|\tilde{f}_u - P_{t_u}\tilde{f}_u| \geq u\}) d(u^q) \leq & \int_0^\infty C' q u^{-1} \int_{\{u \leq |f| \leq cu\}} \Gamma(f)^{p/2} d\mu du \\
\leq & C' q \int_{\mathbb{M}} \Gamma(f)^{p/2} \int_{|f|/c}^{|f|} \frac{du}{u} \, d\mu \\
= & C' q \ln c \int_{\mathbb{M}} \Gamma(f)^{p/2} d\mu.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|f - \tilde{f}_u| = & |f - \tilde{f}_u| \; 1_{\{|f| \leq cu\}} + |f - \tilde{f}_u| \; 1_{\{|f| > cu\}} \\
& = \min(u, |f|) \; 1_{\{|f| \leq cu\}} + (|f| - (c-1)u) \; 1_{\{|f| > cu\}} \leq u + |f| \; 1_{\{|f| > cu\}}.
\end{aligned}$$

By integrating, we obtain then

$$\begin{aligned}
& \int_0^\infty \mu(\{P_{t_u}(|f - \tilde{f}_u|) \geq 2u\}) d(u^q) \leq \int_0^\infty \mu(\{P_{t_u}(|f| \; 1_{\{|f| > cu\}}) \geq u\}) d(u^q) \\
& \leq \int_0^\infty \frac{1}{u} \left(\int_{\mathbb{M}} (|f| \; 1_{\{|f| > cu\}}) d\mu \right) d(u^q) \quad (P_t \text{ is a contraction on } L^1(\mathbb{M})) \\
& = \frac{q}{q-1} \int_{\mathbb{M}} |f| \left(\int_0^\infty 1_{\{|f| > cu\}} d(u^{q-1}) \right) d\mu \\
& = \frac{q}{q-1} \; \frac{1}{c^{q-1}} \|f\|_q^q.
\end{aligned}$$

Gathering all the estimates, we can then conclude

$$\begin{aligned}
\frac{1}{5^q} \int_{\mathbb{M}} |f|^q d\mu = & \frac{1}{5^q} \|f\|_q^q = \int_0^\infty \mu(\{|f| \geq 5u\}) d(u^q) \\
\leq & C' q \ln c \int_{\mathbb{M}} \Gamma(f)^{p/2} d\mu + \frac{q}{q-1} \; \frac{1}{c^{q-1}} \|f\|_q^q
\end{aligned}$$

If we pick a large $c \geq 5$ depending on q such that $\frac{1}{5^q} > \frac{q}{q-1} \; \frac{1}{c^{q-1}}$, we have proved the claim

$$\|f\|_q^q \leq C^q \|\sqrt{\Gamma(f)}\|^p$$

with $C = \left(\frac{C'q \ln c}{\frac{1}{5^q} - \frac{q}{(q-1)c^{q-1}}} \right)^{1/q}$.

Step 3. Finally, it remains to prove $\|f\|_q < \infty$ is actually a consequence of $\|\sqrt{\Gamma(f)}\|_p < \infty$, $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \le 1$, so that we can remove the condition $f \in L^q(\mathbb{M})$ from Step 2 and complete the proof of theorem. From the weak type inequality of Step 1, we have $\|f\|_{q,\infty} < \infty$. For any $0 < \epsilon < 1$, we define

$$N_\epsilon(f) = \int_\epsilon^{1/\epsilon} \mu(\{|f| \ge 5u\}) d(u^q) \le \frac{2q}{5^q} \left(\ln \frac{1}{\epsilon} \right) \|f\|^q_{q,\infty} < \infty.$$

Following the argument in Step 2 again, we see that

$$N_\epsilon(f) \le C'q \ln c \int_{\mathbb{M}} \Gamma(f)^{p/2} d\mu + \int_\epsilon^{1/\epsilon} \frac{1}{u} \left(\int_{\mathbb{M}} (|f| \; 1_{\{|f|>cu\}}) d\mu \right) d(u^q).$$

The first term is bounded, and the second term can be estimated as follows.

$$\begin{aligned}
&\int_\epsilon^{1/\epsilon} \frac{1}{u} \left(\int_{\mathbb{M}} (|f| \; 1_{\{|f|>cu\}}) d\mu \right) d(u^q) \\
=&\int_\epsilon^{1/\epsilon} \frac{1}{u} \left(cu\mu(\{|f| > cu\}) + c \int_u^\infty \mu(\{|f| > cv\}) dv \right) d(u^q) \\
\le&(c + \frac{c}{q-1}) \int_\epsilon^{1/\epsilon} \mu(\{|f| \ge cu\}) d(u^q) + \frac{cq}{(q-1)\epsilon^{q-1}} \int_{1/\epsilon}^\infty \mu(\{|f| \ge cu\}) du \\
\le&\frac{q}{q-1} \frac{5^q}{c^{q-1}} N_\epsilon(f) + \frac{cq}{q-1} \int_{5/c\epsilon}^{1/\epsilon} \frac{\|f\|^q_{q,\infty}}{(cu)^q} d(u^q) + \frac{cq}{(q-1)\epsilon^{q-1}} \int_{1/\epsilon}^\infty \frac{\|f\|^q_{q,\infty}}{(cu)^q} du \\
=&\frac{q}{q-1} \frac{5^q}{c^{q-1}} N_\epsilon(f) + \frac{q}{q-1} \frac{1}{c^{q-1}} \|f\|^q_{q,\infty} \left(q \ln \frac{c}{5} + \frac{1}{q-1} \right)
\end{aligned}$$

So, by choosing c large enough, we have $\sup_{0<\epsilon<1} N_\epsilon(f) < \infty$ which implies $\|f\|_q = \lim_{\epsilon \to 0} 5(N_\epsilon(f))^{1/q} < \infty$. This completes the proof.

■

2.2.4 Sobolev inequality, Isoperimetry and volume growth

In this section, we study the Sobolev and isoperimetric inequalities and their connections with the volume growth of metric balls. We obtain the sub-Riemannian analogue of a theorem essentially due to Ledoux [48].

We first remind what we mean by the perimeter of a set in our subelliptic setting. For further details, we refer to [29].

Let us first observe that, given any point $x \in \mathbb{M}$ there exists an open set $x \in U \subset \mathbb{M}$ in which the operator L can be written as

$$L = -\sum_{i=1}^{m} X_i^* X_i, \tag{2.12}$$

where the vector fields X_i have Lipschitz continuous coefficients in U, and X_i^* indicates the formal adjoint of X_i in $L^2(\mathbb{M}, d\mu)$.

We indicate with $\mathcal{F}(\mathbb{M})$ the set of C^1 vector fields which are subunit for L. Given a function $f \in L^1_{loc}(\mathbb{M})$, which is supported in U we define the horizontal total variation of f as

$$\mathrm{Var}(f) = \sup_{\phi \in \mathcal{F}(\mathbb{M})} \int_U f \left(\sum_{i=1}^{m} X_i^* \phi_i \right) d\mu,$$

where on U, $\phi = \sum_{i=1}^m \phi_i X_i$. For functions not supported in U, $\mathrm{Var}(f)$ may be defined by using a partition of unity. The space

$$BV(\mathbb{M}) = \{ f \in L^1(\mathbb{M}) \mid \mathrm{Var}(f) < \infty \},$$

endowed with the norm

$$||f||_{BV(\mathbb{M})} = ||f||_{L^1(\mathbb{M})} + \mathrm{Var}(f),$$

is a Banach space. It is well-known that $W^{1,1}(\mathbb{M}) = \{ f \in L^1(\mathbb{M}) \mid \sqrt{\Gamma f} \in L^1(\mathbb{M}) \}$ is a strict subspace of $BV(\mathbb{M})$ and when $f \in W^{1,1}(\mathbb{M})$ one has in fact

$$\mathrm{Var}(f) = ||\sqrt{\Gamma(f)}||_{L^1(\mathbb{M})}.$$

Given a measurable set $E \subset \mathbb{M}$ we say that it has finite perimeter, or is a Cacciopoli set if $\mathbf{1}_E \in BV(\mathbb{M})$. In such case the perimeter of E is by definition

$$P(E) = \mathrm{Var}(\mathbf{1}_E).$$

In a later section, we will need the following approximation result, see Theorem 1.14 in [29].

Lemma 2.2.3 *Let $f \in BV(\mathbb{M})$, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in $C_0^\infty(\mathbb{M})$ such that:*

(i) $||f_n - f||_{L^1(\mathbb{M})} \to 0$;

(ii) $\int_{\mathbb{M}} \sqrt{\Gamma(f_n)} d\mu \to \mathrm{Var}(f)$.

We now prove the main result of this subsection.

Theorem 2.2.4 *Let $D > 1$. Let us assume that $\mathbb{M}$ is not compact in the metric topology, then the following assertions are equivalent:*

(1) There exists a constant $C_1 > 0$ such that for every $x \in \mathbb{M}$, $r \geq 0$,

$$\mu(B(x,r)) \geq C_1 r^D.$$

(2) There exists a constant $C_2 > 0$ such that for $x \in \mathbb{M}$, $t > 0$,

$$p(x,x,t) \leq \frac{C_2}{t^{\frac{D}{2}}}.$$

(3) For some $1 \leq p, q, r < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$, there exists a constant $C_3 > 0$ such that $\forall f \in C_0^\infty(\mathbb{M})$, we have

$$\|f\|_q \leq C_3 \| \sqrt{\Gamma(f)} \|_p^{p/q} \ \|f\|_r^{1-p/q}.$$

(4) There exists a constant $C_4 > 0$ such that for every Caccioppoli set $E \subset \mathbb{M}$ one has

$$\mu(E)^{\frac{D-1}{D}} \leq C_4 P(E).$$

Remark 2.2.5 *if we replace the condition of (3) by for all $1 \leq p, q, r < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$, (1),(2), (3) and (4) would still be equivalent.*

Proof That (1) $\to$ (2) follows immediately from the Li-Yau Gaussian upper bound

$$p(x,x,t) \leq \frac{C}{\mu(B(x,\sqrt{t})}$$

that is proved in [10].

The proof that (2) $\to$ (3) follows from the improved Sobolev embedding Theorem 2.2.2.

Indeed, (2) implies first that for $x, y \in \mathbb{M}$,

$$\begin{aligned} p(x,y,t) &= \int_{\mathbb{M}} p(x,z,t/2)p(z,y,t/2)\mu(dy) \\ &\le \sqrt{\int_{\mathbb{M}} p(x,z,t/2)^2\mu(dz)}\sqrt{\int_{\mathbb{M}} p(y,z,t/2)^2\mu(dz)} \\ &= \sqrt{p(x,x,t)p(y,y,t)} \\ &\le \frac{C_2}{t^{\frac{D}{2}}}. \end{aligned}$$

Therefore, for every $f \in L^1(\mathbb{M})$, we have

$$\|P_t(f)\|_\infty = \left\| \int_{\mathbb{M}} p(\cdot,y,t)f(y)\mu(dy) \right\|_\infty \le \|p(\cdot,y,t)\|_\infty \|f\|_1 \le \frac{C_2}{t^{D/2}}\|f\|_1.$$

On the other hand, P_t is a contraction on $L^\infty(\mathbb{M})$, i.e. $\|P_t\|_{\infty\to\infty} \le 1$. Therefore, by the Riesz-Thorin interpolation theorem, we deduce that we have the following heat semigroup embedding

$$\|P_t\|_{r\to\infty} \le \frac{C_2^{1/r}}{t^{D/2r}}, \quad r \ge 1.$$

Let now $1 \le p, q, r < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$. Since for $\theta = \frac{p}{q}$, $-\frac{\theta}{2(\theta-1)} - \frac{D}{2r} = 0$, we have

$$\begin{aligned} \|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} &= \sup_{t>0} t^{-\theta/2(\theta-1)}\|P_t f\|_\infty \\ &\le \sup_{t>0} t^{-\theta/2(\theta-1)} \frac{C_2^{1/r}}{t^{D/2r}}\|f\|_r = C_2^{1/r}\|f\|_r, \end{aligned}$$

we can conclude (3) from the improved Sobolev embeddding of Theorem 2.2.2.

The proof that (3) is equivalent to (4) follows the classical ideas of Fleming-Rishel and Maz'ya, and it is based on a generalization of Federer's co-area formula for the space $BV(\mathbb{M})$, see for instance [29].

Finally, we show that (3) $\to$ (1). We adapt an idea in [58] (see Theorem 3.1.5 on p. 58). For any fix $x \in \mathbb{M}$, $s > 0$, consider the function

$$f(y) = \max\{s - d(x,y), 0\}.$$

Then, it is easily seen that

$$\begin{aligned}\|f\|_q \geq &(s/2)\mu(B(x,s/2))^{1/q}\\ \|f\|_r \leq &s\mu(B(x,s))^{1/r}\\ \|\sqrt{\Gamma(f)}\|_p \leq &\mu(B(x,s))^{1/p}.\end{aligned}$$

Hence, from (3) we have

$$\begin{aligned}\mu(B(x,s/2))^{1/q} \leq &2C_3 s^{-p/q}\mu(B(x,s))^{1/q+(1/r)(1-p/q)}\\ = &2C_3 s^{-p/q}\mu(B(x,s))^{1/q+p/qD}.\end{aligned}$$

This can be written as follows.

$$\mu(B(x,s)) \geq (2C_3)^{-Dq/(D+p)} s^{Dp/(D+p)} \mu(B(x,s/2))^{D/(D+p)}.$$

$$\mu(B(x,s)) \geq \{(2C_3)^{-q} s^p\}^a \mu(B(x,s/2))^a$$

where $a = D/(D+p) < 1$. Replacing s by $s/2$ iteratively, we obtain

$$\mu(B(x,s)) \geq (2C_3)^{-q(\sum_{j=1}^i a^j)} s^{p(\sum_{j=1}^i a^j)} 2^{-p(\sum_{j=1}^i (j-1)a^j)} \mu(B(x,s/2^i))^{a^i}.$$

From the volume doubling property proved in [8], we have the control

$$\mu(B(x,s/2^i)) \geq C^{-1}(1/2^i)^Q \mu(B(x,s)),$$

for some $C = C(\rho_1, \rho_2, \kappa, d) > 0$ and $Q = \log_2 C$.

Therefore, we have

$$\liminf_{i\to\infty} \mu(B(x,s/2^i))^{a^i} \geq \lim_{i\to\infty} (C^{-1}\mu(B(x,s)))^{a^i} (1/2)^{iQa^i} = 1.$$

Since $\sum_{j=1}^\infty a^j = D/p$, $\sum_{j=1}^\infty (j-1)a^j = D^2/p^2$, we obtain the volume growth control

$$\mu(B(x,s)) \geq 2^{-(q+D)D/p} C_3^{-qD/p} s^D.$$

This establishes (1), thus completing the proof.

■

Remark 2.2.6 *By combining the results of [8] and [29], an alternative proof of (1) → (4) could be given. Indeed, in [29] it was proved that in a Carnot-Carathéodory space* (X, μ, d) *the doubling condition*

$$\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)), \quad x \in X, r > 0,$$

for the volume of the metric balls combined with a weak Poincaré inequality suffice to establish the following basic relative isoperimetric inequality

$$\begin{aligned} \min\{\mu(E \cap B(x, r)), \mu((X \setminus E) \cap B(x, r))\}^{\frac{D-1}{D}} \\ \leq C_{iso} \left(\frac{r^D}{\mu(B(x, r))}\right)^{\frac{1}{D}} P(E, B(x, r)), \end{aligned} \tag{2.13}$$

where $E \subset X$ *is any set of locally finite perimeter. In this inequality the number* $D = \log_2 C_1$*, where* C_1 *is the doubling constant, and* C_{iso} *is a constant which depends only on* C_1 *and on the constant in the Poincaré inequality. If in addition the space* X *satisfies the volume growth condition*

$$\mu(B(x, r)) \geq C_2 r^D, \quad x \in \mathbb{M}, r > 0, \tag{2.14}$$

then (2.13) *gives the global isoperimetric inequality*

$$\mu(E)^{\frac{D-1}{D}} \leq C_{iso} P(E, \mathbb{M}), \tag{2.15}$$

for any measurable set of locally finite perimeter $E \subset \mathbb{M}$*. Since in [8], it was proved that the doubling condition and the weak Poincaré inequality are satisfied when* $\rho_1 \geq 0$*, we conclude that (1) → (4).*

2.3 The case $\rho_1 > 0$

Throughout this Section 3, we assume that L satisfies the *generalized curvature dimension inequality* $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 > 0$, $\rho_2 > 0$ and $\kappa \geq 0$. The following gradient bound was also proved in [10].

Theorem 2.3.1 (Li-Yau type gradient estimate with $\rho_1 > 0$) *Let $f \in C_0^\infty(\mathbb{M})$, $f \geq 0$, $f \not\equiv 0$, then the following inequality holds for $t > 0$:*

$$\Gamma(\ln P_t f) \leq \frac{2\rho_2 + 3\kappa}{2\rho_2} e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t} \frac{LP_t f}{P_t f} + \frac{d\rho_1}{12\rho_2} \frac{(2\rho_2 + 3\kappa)^2}{\rho_2 + \kappa} \frac{e^{-\frac{4\rho_1\rho_2}{3(\rho_2+\kappa)}t}}{1 - e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t}}. \tag{2.16}$$

2.3.1 Gradient bounds for the heat semigroup

We first establish the following reverse Poincaré inequality.

Proposition 2.3.1 *For $f \in C_0^\infty(\mathbb{M})$ and $t \geq 0$,*

$$\Gamma(P_t f) \leq \frac{1}{2}\rho_1 \frac{\rho_2 + 2\kappa}{\rho_2 + \kappa} \frac{e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}{1 - e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}} (P_t f^2 - (P_t f)^2).$$

Proof Let us fix $T > 0$ once time for all in the following proof. Given a function $f \in C_0^\infty(\mathbb{M})$, for $0 \leq t \leq T$ we introduce the functionals

$$\phi_1(x,t) = \Gamma(P_{T-t}f)(x),$$

and

$$\phi_2(x,t) = \Gamma^Z(P_{T-t}f)(x),$$

which are defined on $\mathbb{M} \times [0,T]$. A straightforward computation shows that

$$L\phi_1 + \frac{\partial \phi_1}{\partial t} = 2\Gamma_2(P_{T-t}f).$$

and

$$L\phi_2 + \frac{\partial \phi_2}{\partial t} = 2\Gamma_2^Z(P_{T-t}f).$$

Consider now the function

$$\begin{aligned}\phi(x,t) &= a(t)\phi_1(x,t) + b(t)\phi_2(x,t)\\ &= a(t)\Gamma(P_{T-t}f)(x) + b(t)\Gamma^Z(P_{T-t}f)(x),\end{aligned}$$

where a and b are two non negative functions that will be chosen later. Applying the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$, we obtain

$$\begin{aligned} L\phi + \frac{\partial \phi}{\partial t} =& a'\Gamma(P_{T-t}f) + b'\Gamma^Z(P_{T-t}f) \\ &+ 2a\Gamma_2(P_{T-t}f) + 2b\Gamma_2^Z(P_{T-t}f) \\ &\geq \left(a' + 2\rho_1 a - 2\kappa\frac{a^2}{b}\right)\Gamma(P_{T-t}f) + (b' + 2\rho_2 a)\Gamma^Z(P_{T-t}f). \end{aligned}$$

Let us now chose

$$b(t) = \left(e^{-\frac{2\rho_1\rho_2 t}{\kappa+\rho_2}} - e^{-\frac{2\rho_1\rho_2 T}{\kappa+\rho_2}}\right)^2$$

and

$$a(t) = -\frac{b'(t)}{2\rho_2},$$

so that

$$b' + 2\rho_2 a = 0$$

and

$$a' + 2\rho_1 a - 2\kappa\frac{a^2}{b} \geq \rho_1\frac{\rho_2 + 2\kappa}{\rho_2 + \kappa}e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}T}.$$

With this choice, we get therefore

$$L\phi + \frac{\partial \phi}{\partial t} \geq -\rho_1\frac{\rho_2 + 2\kappa}{\rho_2 + \kappa}e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}T}\Gamma(P_{T-t}f).$$

and therefore from a comparison theorem for parabolic partial differential equations (see [10]) we have

$$P_T(\phi(\cdot, T))(x) \geq \phi(x, 0) - \rho_1\frac{\rho_2 + 2\kappa}{\rho_2 + \kappa}e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}T}\int_0^T P_t(\Gamma(P_{T-t}f))dt.$$

It is easily seen that

$$\int_0^T P_t(\Gamma(P_{T-t}f))dt = \frac{1}{2}(P_T f^2 - (P_T f)^2),$$

and since,

$$\phi(x, 0) = a(0)\Gamma(P_T f)(x) + b(0)\Gamma^Z(P_T f)(x)$$

and

$$P_T(\phi(\cdot,T))(x) = a(T)P_T(\Gamma(f))(x) + b(T)P_T(\Gamma^Z(f))(x) = 0,$$

the proof is completed. ∎

Proposition 2.3.2 *Let $f \in C_0^\infty(\mathbb{M})$.*

- *If $1 \le p < 2$, then for every $t > 0$,*

$$\left\| \sqrt{\Gamma(P_t f)} \right\|_p \le \left(\frac{\frac{d\rho_1\rho_2}{3(\rho_2+\kappa)} \frac{(1+\frac{3\kappa}{2\rho_2})^2 e^{-\frac{4\rho_1\rho_2}{3(\rho_2+\kappa)}t}}{(1-e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t})}}{1+(p-1)(1+\frac{3\kappa}{2\rho_2})e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t}} \right)^{\frac{1}{2}} \|f\|_p.$$

- *If $2 \le p \le +\infty$, then for every $t > 0$,*

$$\left\| \sqrt{\Gamma(P_t f)} \right\|_p \le \left(\frac{1}{2}\rho_1 \frac{\rho_2+2\kappa}{\rho_2+\kappa} \frac{e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}} \right)^{\frac{1}{2}} \|f\|_p.$$

Proof The proof is essentially identical to the proof of Proposition 2.2.1. We observe from this proof that if for $f \in C_0^\infty(\mathbb{M}), t > 0$

$$\Gamma(\ln P_t f) \le \alpha(t)\frac{LP_t f}{P_t f} + \beta(t), \qquad f \ge 0, f \not\equiv 0, \alpha(t), \beta(t) > 0$$

$$\Gamma(P_t f) \le \gamma(t)(P_t f^2 - (P_t f)^2), \qquad \gamma(t) > 0$$

then

$$\|\sqrt{\Gamma(P_t f)}\|_p \le \left(\frac{\beta(t)}{1+(p-1)\alpha(t)}\right)^{\frac{1}{2}} \|f\|_p, \quad \text{for } 1 \le p < 2$$

$$\|\sqrt{\Gamma(P_t f)}\|_p \le (\gamma(t))^{\frac{1}{2}} \|f\|_p, \qquad \text{for } 2 \le p < \infty.$$

By Theorem 2.3.1 and Proposition 2.3.1, we then see that $\alpha(t), \beta(t), \gamma(t)$ are given by:

$$\alpha(t) = (1+\frac{3\kappa}{2\rho_2})e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t}, \quad \beta(t) = \frac{d\rho_1\rho_2}{3(\rho_2+\kappa)} \frac{(1+\frac{3\kappa}{2\rho_2})^2 e^{-\frac{4\rho_1\rho_2}{3(\rho_2+\kappa)}t}}{(1-e^{-\frac{2\rho_1\rho_2}{3(\rho_2+\kappa)}t})}$$

$$\gamma(t) = \frac{1}{2}\rho_1 \frac{\rho_2+2\kappa}{\rho_2+\kappa} \frac{e^{-2\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}.$$

∎

2.3.2 Pseudo-Poincaré inequalities

Proposition 2.3.3 *Let $f \in C_0^\infty(\mathbb{M})$.*

- *If $1 \le p < 2$, then for every $t \ge 0$,*

$$\|f - P_t f\|_p \le \left(\frac{2(\rho_2 + 2\kappa)(\rho_2 + \kappa)}{\rho_1 \rho_2^2} (1 - e^{-\frac{\rho_1 \rho_2}{\rho_2 + \kappa} t}) \right)^{\frac{1}{2}} \|\sqrt{\Gamma(f)}\|_p. \tag{2.17}$$

- *If $2 \le p \le +\infty$, then for every $t \ge 0$,*

$$\|f - P_t f\|_p \le (1 + \frac{3\kappa}{2\rho_2}) \left(\frac{3d(\rho_2 + \kappa)}{\rho_1 \rho_2} (1 - e^{-\frac{2\rho_1 \rho_2}{3(\rho_2 + \kappa)} t}) \right)^{\frac{1}{2}} \|\sqrt{\Gamma(f)}\|_p. \tag{2.18}$$

Proof As shown in the proof of Proposition 2.2.2, we have

$$\|f - P_t f\|_p \le \left(\int_0^t \sqrt{\gamma(s)} ds \right) \|\sqrt{\Gamma(f)}\|_p, \quad \text{for } 1 \le p < 2$$

$$\|f - P_t f\|_p \le \left(\int_0^t \sqrt{\frac{\beta(s)}{1 + (p-1)\alpha(s)}} ds \right) \|\sqrt{\Gamma(f)}\|_p, \quad \text{for } 2 \le p < \infty$$

where α, β, γ are defined in the proof of Proposition 2.3.2. The proof is done by

$$\begin{aligned} \int_0^t \sqrt{\gamma(s)} ds &= \int_0^t \left(\frac{\rho_1(\rho_2 + 2\kappa)}{2(\rho_2 + \kappa)} \right)^{\frac{1}{2}} \frac{e^{-\frac{\rho_1 \rho_2}{\rho_2 + \kappa} s}}{\sqrt{1 - e^{-\frac{\rho_1 \rho_2}{\rho_2 + \kappa} s}}} ds \\ &= \left(\frac{2(\rho_2 + 2\kappa)(\rho_2 + \kappa)}{\rho_1 \rho_2^2} (1 - e^{-\frac{\rho_1 \rho_2}{\rho_2 + \kappa} t}) \right)^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \int_0^t \sqrt{\frac{\beta(s)}{1 + (p-1)\alpha(s)}} ds &\le \int_0^t \sqrt{\beta(s)} ds \\ &= \int_0^t \left(\frac{d\rho_1 \rho_2}{3(\rho_2 + \kappa)} \frac{(1 + \frac{3\kappa}{2\rho_2})^2 e^{-\frac{4\rho_1 \rho_2}{3(\rho_2 + \kappa)} s}}{(1 - e^{-\frac{2\rho_1 \rho_2}{3(\rho_2 + \kappa)} s})} \right)^{\frac{1}{2}} ds \\ &= \left(1 + \frac{3\kappa}{2\rho_2} \right) \left(\frac{3d(\rho_2 + \kappa)}{\rho_1 \rho_2} (1 - e^{-\frac{2\rho_1 \rho_2}{3(\rho_2 + \kappa)} t}) \right)^{\frac{1}{2}}. \end{aligned}$$

■

2.3.3 Poincaré inequality

In the case of $\rho_1 > 0$, we have the following theorem which is proved in [7].

Theorem 2.3.2 *The measure μ is finite, i.e. $\mu(\mathbb{M}) < +\infty$ and for every $1 \leq p \leq \infty$, $f \in L^p(\mathbb{M})$,*

$$P_t f \xrightarrow{t\to\infty} \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu.$$

This theorem allows us to deduce the Poincaré inequality.

Proposition 2.3.4 *Let $1 \leq p < \infty$. There exists $C = C_p(\rho_1, \rho_2, \kappa, d) > 0$ such that, for $\forall f \in C_0^\infty(\mathbb{M})$,*

$$\|f - f_{\mathbb{M}}\|_p \leq C \|\sqrt{\Gamma(f)}\|_p,$$

where $f_{\mathbb{M}} = \frac{1}{\mu(\mathbb{M})} \int_{\mathbb{M}} f d\mu$.

Proof The proof is immediate from Proposition 2.3.3 and Theorem 2.3.2 by letting $t \to \infty$. And C is given by

$$C_p(\rho_1, \rho_2, \kappa, d) = \begin{cases} \left(\frac{2(\rho_2+2\kappa)(\rho_2+\kappa)}{\rho_1 \rho_2^2}\right)^{\frac{1}{2}} & \text{if } 1 \leq p < 2, \\ (1 + \frac{3\kappa}{2\rho_2}) \left(\frac{3d(\rho_2+\kappa)}{\rho_1\rho_2}\right)^{\frac{1}{2}} & \text{if } 2 \leq p < \infty. \end{cases}$$

■

2.3.4 A lower bound on the Cheeger's isoperimetric constant

In [20], in order to bound from below the first eigenvalue λ_1 of a compact Riemannian manifold with normalized Riemannian measure μ_g, Cheeger's introduced the following isoperimetric constant

$$h = \inf \frac{\mu_g(\partial A)}{\mu_g(A)},$$

where the infimum runs over all open subsets A with smooth boundary ∂A such that $\mu(A) \leq \frac{1}{2}$. Cheeger's inequality then writes $\lambda_1 \geq \frac{h^2}{4}$.

Such isoperimetric quantity may also be considered and estimated in our sub-Riemannian framework. Throughout this section, we assume $\mu(\mathbb{M})=1$. Let

$$\iota = \inf \frac{P(E)}{\mu(E)}$$

where the infimum runs over all Caccioppoli sets E such that $\mu(E) \le \frac{1}{2}$ (we remind that $P(E)$ denotes the perimeter of E as defined in Section 2.4). By following the argument of Ledoux in [47] we see that $\lambda_1 \ge \frac{i^2}{4}$ where λ_1 is the first eigenvalue of $-L$. The next proposition gives a lower bound on ι (and therefore on λ_1).

Proposition 2.3.5 *Let $E \subset \mathbb{M}$ be a Caccioppoli set. We have*

$$\mu(E)(1-\mu(E)) \le \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right) P(E).$$

As a consequence

$$\iota \ge \frac{1}{2}\sqrt{\frac{\rho_1}{2}}\frac{1}{1+\frac{2\kappa}{\rho_2}}.$$

Proof We know from the pseudo-Poincaré inequality that for $f \in C_0^\infty(\mathbb{M})$,

$$\|P_t f - f\|_1 \le \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right)\sqrt{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}\|\sqrt{\Gamma(f)}\|_1, \quad t>0. \tag{2.19}$$

Suppose now that $E \subset \mathbb{M}$ is a Caccioppoli set. By Proposition 2.2.3 there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{M})$ satisfying (i) and (ii) of that Proposition. Applying (2.19) to f_n, we obtain

$$\begin{aligned}\|P_t f_n - f_n\|_1 \le& \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right)\sqrt{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}\|\sqrt{\Gamma(f_n)}\|_1 \\ =& \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right)\sqrt{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}\,\mathrm{Var}(f_n).\end{aligned}$$

Letting $n\to\infty$ in this inequality, we conclude

$$\begin{aligned}\|P_t\mathbf{1}_E - \mathbf{1}_E\|_{L^1(\mathbb{M})} \le& \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right)\sqrt{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}\,\mathrm{Var}(\mathbf{1}_E) \\ =& \sqrt{\frac{2}{\rho_1}}\left(1+\frac{2\kappa}{\rho_2}\right)\sqrt{1-e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}P(E).\end{aligned}$$

Observe now that, using $P_t 1 = 1$, we have

$$\begin{aligned}
\|P_t\mathbf{1}_E - \mathbf{1}_E\|_{L^1(\mathbb{M})} \geq & \int_{\mathbb{M}} |\mathbf{1}_{E^c}| \ \ |P_t\mathbf{1}_E - \mathbf{1}_E| d\mu \\
& \geq \int_{\mathbb{M}} \mathbf{1}_{E^c}(P_t\mathbf{1}_E - \mathbf{1}_E) d\mu = \int_{\mathbb{M}} \mathbf{1}_{E^c} P_t \mathbf{1}_E d\mu \\
& = \int_{\mathbb{M}} P_t\mathbf{1}_E d\mu - \int_{\mathbb{M}} \mathbf{1}_E P_t \mathbf{1}_E d\mu = \int_{\mathbb{M}} \mathbf{1}_E d\mu - \int_E P_t\mathbf{1}_E d\mu \\
& = \mu(E) - \int_E P_t\mathbf{1}_E d\mu
\end{aligned}$$

On the other hand, from the semigroup property we have

$$\int_E P_t\mathbf{1}_E d\mu = \int_{\mathbb{M}} \left(P_{t/2}\mathbf{1}_E\right)^2 d\mu.$$

We thus obtain

$$||P_t\mathbf{1}_E - \mathbf{1}_E||_{L^1(\mathbb{M})} \geq \left(\mu(E) - \int_{\mathbb{M}} \left(P_{t/2}\mathbf{1}_E\right)^2 d\mu\right).$$

In [10], it has been proved that for $x, y \in \mathbb{M}$ and $t > 0$,

$$p(x,y,t) \leq \frac{1}{\left(1 - e^{-\frac{2\rho_1\rho_2 t}{3(\rho_2+\kappa)}}\right)^{\frac{d}{2}\left(1+\frac{3\kappa}{2\rho_2}\right)}}.$$

This gives

$$\begin{aligned}
\int_{\mathbb{M}} (P_{t/2}\mathbf{1}_E)^2 d\mu \leq & \left(\int_E \left(\int_{\mathbb{M}} p(x,y,t/2)^2 d\mu(y)\right)^{\frac{1}{2}} d\mu(x)\right)^2 \\
& = \left(\int_E p(x,x,t)^{\frac{1}{2}} d\mu(x)\right)^2 \leq \frac{1}{\left(1 - e^{-\frac{2\rho_1\rho_2 t}{3(\rho_2+\kappa)}}\right)^{d\left(1+\frac{3\kappa}{2\rho_2}\right)}} \mu(E)^2.
\end{aligned}$$

Combining these equations we reach the conclusion

$$\sqrt{\frac{2}{\rho_1}}\left(1 + \frac{2\kappa}{\rho_2}\right)\sqrt{1 - e^{-\frac{\rho_1\rho_2}{\rho_2+\kappa}t}}\, P(E) \geq \mu(E) - \frac{1}{\left(1 - e^{-\frac{2\rho_1\rho_2 t}{3(\rho_2+\kappa)}}\right)^{d\left(1+\frac{3\kappa}{2\rho_2}\right)}} \mu(E)^2.$$

We conclude by letting $t \to +\infty$. ∎

2.3.5 A Lichnerowicz type theorem

A well-known theorem of Lichnerowicz asserts that on a d-dimensional complete Riemannian manifold whose Ricci curvature is bounded below by a non negative constant ρ, then the first eigenvalue of the Laplace-Beltrami operator is bounded below by $\frac{\rho d}{d-1}$. In this section, we provide a similar theorem for our operator L. Let us observe that in [31], Greenleaf obtained a similar result for the sub-Laplacian on a CR manifold. A recent work of Hladky [34] also gives lower bounds for the first eigenvalue of sub-Laplacians on some sub-Riemannian manifolds.

Proposition 2.3.6 *The first non zero eigenvalue λ_1 of $-L$ satisfies the estimate*

$$\lambda_1 \geq \frac{\rho_1 \rho_2}{\frac{d-1}{d}\rho_2 + \kappa}.$$

Proof Let $f : \mathbb{M} \to \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue $-\lambda_1$. From the generalized curvature dimension inequality we know that for every $\nu > 0$,

$$\Gamma_2(f,f) + \nu \Gamma_2^Z(f,f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f,f) + \rho_2 \Gamma^Z(f,f).$$

By integrating this inequality on the manifold $\mathbb{M}$, we obtain

$$\begin{aligned}\int_{\mathbb{M}} \Gamma_2(f,f)d\mu + \nu \int_{\mathbb{M}} \Gamma_2^Z(f,f)d\mu& \\ \geq \frac{1}{d}\int_{\mathbb{M}} (Lf)^2 d\mu + \left(\rho_1 - \frac{\kappa}{\nu}\right)&\int_{\mathbb{M}} \Gamma(f,f)d\mu + \rho_2 \int_{\mathbb{M}} \Gamma^Z(f,f)d\mu.\end{aligned}$$

Let us now recall that

$$\Gamma_2(f,f) = \frac{1}{2}\big[L\Gamma(f,f) - 2\Gamma(f,Lf)\big],$$

and

$$\Gamma_2^Z(f,f) = \frac{1}{2}\big[L\Gamma^Z(f,f) - 2\Gamma^Z(f,Lf)\big].$$

Therefore, by using $Lf = -\lambda_1 f$ and integrating by parts in the above inequality, we find

$$\left(\lambda_1^2 - \frac{\lambda_1^2}{d} + \frac{\kappa\lambda_1}{\nu} - \rho_1\lambda_1\right)\int_{\mathbb{M}} f^2 d\mu \geq (\rho_2 - \nu\lambda_1)\int_{\mathbb{M}} \Gamma^Z(f,f)d\mu.$$

By choosing $\nu = \frac{\rho_2}{\lambda_1}$, we obtain the inequality

$$\lambda_1 \geq \frac{\rho_1 \rho_2}{\frac{d-1}{d}\rho_2 + \kappa}.$$

■

Remark 2.3.3 *We note that when $\kappa = 0$, which corresponds to the Riemannian case, we recover the classical theorem of Lichnerowicz.*

3. POINCARÉ INEQUALITIES ON BALLS

3.1 Introduction

Let $\mathbb{M}$ be a sub-Riemannian manifold in [10] satisfying the generalized curvature dimension inequalities $CD(-K,\rho_2,\kappa,d), K > 0$ introduced by F.Baudoin and N.Garofalo. (discuss [14] for the enlarged framework - Riemannian foliations). $\mathbb{M}$ is a sub-Riemannian analogue of a Riemannian manifold with Ricci curvature bounded below by $-K$. Instead of the Riemannian measure and Laplace-Beltrami operator, $\mathbb{M}$ is equipped with a smooth positive measure μ and the sub-Laplacian L which is subelliptic of order 1/2. Our class of $\mathbb{M}$ contains CR contact (or quaternionic contact) Sasakian manifolds with Tanaka-Webster-Ricci curvature bounded below by $-K$ and Carnot group of step 2 (find more examples and applications in [10], [14], [8], [9], [7]).

Our first goal in this chapter is to prove the following scale-invariant Poincaré and Sobolev inequalities on Carnot-Carathéodory balls $B = B(x_0, r) \subset \mathbb{M}, \forall r \in (0,\infty), x_0 \in \mathbb{M}$. For $f \in C^\infty(\overline{B}), g \in C_0^\infty(\overline{B})$,

$$\int_B |f - f_B|^2 d\mu \le r^2 e^{C_p(1+Kr^2)} \int_B \Gamma(f) d\mu, \tag{3.1}$$

$$\left(\frac{1}{\mu(B)} \int_B |g|^{\frac{2Q}{Q-2}} d\mu\right)^{\frac{Q-2}{2Q}} \le r e^{C_s(1+Kr^2)} \left(\frac{1}{\mu(B)} \int_B (\Gamma(g) + r^{-2}|g|^2) d\mu\right)^{\frac{1}{2}}, \tag{3.2}$$

where $f_B = \frac{1}{\mu(B)} \int_B f d\mu$, $Q = \log_2 C_{d1}$ in (3.10) is a "homogeneous dimension" of $\mathbb{M}$, $C_p, C_s > 0$ depend on ρ_2, κ, d, and $\Gamma(f) = \frac{1}{2}L(f^2) - fLf$ is the Bakry-Émery's *carré du champ* (square of the gradient).

On Riemannian manifolds with Ricci curvature bounded from below by $-K < 0$, Buser proved the upper bound for the Cheeger isoperimetric constant, $h(B(x,r)) \le e^{-C(1+K\sqrt{r})} r^{-1}, C > 0$ in Lemma 5.1, [19]. With the well-known Cheeger's inequality

$\lambda_1(B) \geq h^2(B)/4$ for the first Neumann eigenvalue on the ball, the Poincaré inequality is implied as follows (see [59], section 9 of [60]):

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq r^2 e^{C(1+\sqrt{K}r)} \int_{B(x,r)} |\nabla f|^2 d\mu, \quad \forall f \in C^\infty(\mathbb{M}), r > 0. \tag{3.3}$$

Unfortunately, Buser's method relies on the Laplacian comparison theorem, which is not established on sub-Riemannian manifolds. (one can find the Laplacian comparison theorem in [1], [45] under additional conditions.)

For a square of vector fields, Jerison proved the Poincaré inequalities on Carnot-Carathéodory balls in [39] as well as Jerison and Sanchez-Calle proved the same inequalities for a general subelliptic operator in [41]. But these are only for balls with small radii.

In case of $\mathbb{M}$ with $CD(0, \rho_2, \kappa, d)$ - nonnegative Ricci curvature, the scale-invariant Poincaré inequality can be found in theorem 4.2, corollary 4.3 of [8]. The underlying theory for the proof is the equivalence between two-sided Gaussian bounds for the heat kernel and the conjunction of the volume doubling property and the Poincaré inequality. To prove our Poincaré inequality in negatively-curved space, even with the other ingredients proved in [9], we need to modify the proof of the Poincaré inequality by Kusuoka-Stroock in [44] to deal with $e^{C(1+Kr^2)}$. Our proof for the following scale-invariant Poincaré inequality is in section 3:

Theorem 3.1.1 (Poincaré inequality) *If $\mathbb{M}$ satisfies $CD(-K, \rho_2, \kappa, d)$ with $K > 0$, for any $r > 0$, $x_0 \in \mathbb{M}$ and $f \in C^\infty(\mathbb{M})$,*

$$\int_{B(x_0,r)} |f(x) - f_{B(x_0,r)}|^2 d\mu(x) \leq r^2 e^{C_p(1+Kr^2)} \int_{B(x_0,r)} \Gamma(f) d\mu, \tag{3.4}$$

where $C_p > 0$ depend on ρ_2, κ, d.

The Sobolev inequality 3.2 is a consequence of the Poincaré inequality by the arguments [59]. In fact, the proof can be directly obtained from theorem 13 in [9], adapting the arguments of section 10 in [60].

Our second goal is to study the uniqueness of solutions of certain types for the following Cauchy problem - the heat equations associated with the sub-Laplacian L :

$$\left(\frac{\partial}{\partial t} - L\right) u(x,t) = 0, \tag{3.5}$$

The initial condition will be given by the type of solutions we want.

Adapting the ideas of [23], [51], the uniqueness theorem is proved as a consequence of our Poincaré, Sobolev inequalities :

Theorem 3.1.2 (Uniqueness of the nonnegative solution) *Assume $CD(-K, \rho_2, \kappa, d)$, $K > 0$ on $\mathbb{M}$. For any $f \in C(\mathbb{M}), f \geq 0$, nonnegative weak solution $u \in C(\mathbb{M} \times [0, \infty))$ for (3.5) with the initial condition $u(x, 0) = f(x)$ is uniquely determined, and $u(x,t) = P_t f(x)$, where $P_t f(x) = \int_{\mathbb{M}} p_t(x,y) f(y) d\mu$ with the heat kernel $p_t(x,y)$ in [9].*

Theorem 3.1.3 (Uniqueness of L^p solutions) *Assume $CD(-K, \rho_2, \kappa, d)$, $K > 0$ on $\mathbb{M}$. If $f \in L^1(\mathbb{M})$ is given, then we have the unique L^1 solution $u(\cdot, t) \in L^1(\mathbb{M}), t \in (0, \infty)$ for (3.5) with the initial data $u \xrightarrow{L^1} f \in L^1(\mathbb{M})$ as $t \to 0$.*
On the other hand, for $1 < p < \infty$, we have the uniqueness of the L^p solution for (3.5) without any curvature condition

Theorem 3.1.2 improves Li-Yau/Harnack inequality for the subelliptic L which is proved for $P_t f$ in [10], [9]. (Find notations in the following section. See Remark 3.2.2 for $\mathcal{A}_\epsilon$.)

Corollary 3.1.4 (Li-Yau type inequality, [10]) *Assume $CD(\rho_1, \rho_2, \kappa, d)$, $\rho_1 \in \mathbb{R}$. Any non-negative solution $u(x,t) \in \mathcal{A}_\epsilon$ of (3.5) satisfies the following inequality:*

$$\Gamma(\ln u) + \frac{2\rho_2}{3} t \Gamma^Z(\ln u) - \left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} t\right) \frac{Lu}{u}$$
$$\leq \frac{d\rho_1^2}{6} t - \frac{d\rho_1}{2}\left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{d}{2t}\left(1 + \frac{3\kappa}{2\rho_2}\right)^2.$$

Corollary 3.1.5 (Harnack inequality, [9], [10]) *Assume $CD(-K,\rho_2,\kappa,d)$, $K \geq 0$. Then any non-negative solution $u \not\equiv 0$ of (3.5) satisfies for $x,y \in \mathbb{M}$, $s < t \in \mathbb{R}_+$,*

$$\frac{u(x,s)}{u(y,t)} \leq \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{dK}{4}(t-s) + \frac{K}{12}d(x,y)^2 + \frac{d(x,y)^2}{4(t-s)}\left(1+\frac{3\kappa}{2\rho_2}\right)\right).$$

3.2 Preliminaries

In this section, we introduce the framework of the sub-Riemannian manifolds satisfying the generalized curvature dimension inequalities. After F.Baudoin and N.Garofalo introduced the generalized curvature dimension inequality in [10], this framework is extended to the Riemannian foliations with totally geodesic leaves by the recent work in [14].

L is a second-order diffusion operator with real C^∞ coefficients on $\mathbb{M}$ which satisfies the subelliptic estimate in the sense of [27]. Also L is symmetric, non-positive and has no zero order term, i.e.:

$$\int_{\mathbb{M}} fLg d\mu = \int_{\mathbb{M}} gLf d\mu, \quad \int_{\mathbb{M}} fLf d\mu \leq 0, \quad L1 = 0,$$

for every $f, g \in C_0^\infty(\mathbb{M})$.

The intrinsic sub-Riemannian metric associated with L is defined by the minimal length of subunit curve:

$$\begin{aligned} d(x,y) = \inf\big\{T \big| &\exists \text{ Lipschitz } \gamma : [0,T] \to \mathbb{M}, \gamma(0) = x, \gamma(T) = y, \\ &|\frac{d}{dt}f(\gamma(t))| \leq \sqrt{\Gamma f(\gamma(t))}, \forall f \in C^\infty(\mathbb{M}), \text{ almost every } t \in [0,T]\big\}, \end{aligned}$$

where $\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf)$, $\Gamma(f) = \Gamma(f,f)$. We assume that the metric space $(\mathbb{M}, d)$ is complete.

Following Strichartz [61], the completeness assumption of $(\mathbb{M}, d)$ yields that L is essentially self-adjoint on $C_0^\infty(\mathbb{M})$. So we can denote by L the unique self-adjoint extension (the Friedrichs extension) of L in $L^2(\mathbb{M}, \mu)$. Maximum principle([18]) and Hörmander's hypoellipticity of L are well-known. See [27], [39], [10], [54] for more properties of L.

We follow the steps in [10] to introduce the curvature assumption on our subelliptic framework. In addition to Γ, we assume that $\mathbb{M}$ is endowed with another smooth symmetric bilinear differential form, indicated with Γ^Z, satisfying for $f, g \in C^\infty(\mathbb{M})$

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),$$

and $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$.

We make the following assumptions that will be in force throughout the article:

(H.1) There exists an increasing sequence $h_k \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on $\mathbb{M}$, and

$$||\Gamma(h_k)||_\infty + ||\Gamma^Z(h_k)||_\infty \to 0, \quad \text{as } k \to \infty.$$

(H.2) For any $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

(H.3) For every $t \geq 0$, $P_t 1 = 1$ and for every $f \in C_0^\infty(\mathbb{M})$ and $T \geq 0$, one has

$$\sup_{t \in [0,T]} \|\Gamma(P_t f)\|_\infty + \|\Gamma^Z(P_t f)\|_\infty < +\infty,$$

where P_t is the heat semigroup generated by L.

(Details about the assumptions are discussed in [10]) The assumption (H.1) is implied by the completeness of the metric space. In the sub-Riemannian geometries covered by the present work, the assumption (H.2) means that the torsion of the sub-Riemannian connection is vertical (for instance, Sasakian condition of CR manifolds). Removing this assumption in certain cases is discussed in [15]. Assumption (H.3) is necessary to rigorously justify the Bakry-Émery type arguments. It is a consequence of the generalized curvature dimension inequality below in many examples (see [10]).

In addition to Γ and Γ^Z, we denote the following second order differential bilinear forms: for any $f, g \in C^\infty(\mathbb{M})$,

$$\Gamma_2(f, g) = \frac{1}{2}\big[L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)\big], \tag{3.6}$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}\big[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)\big]. \tag{3.7}$$

As for Γ and Γ^Z, we denote $\Gamma_2(f) = \Gamma_2(f,f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

The following curvature dimension condition was introduced in [10]. See also Theorem 5.1 in [14].

Definition 3.2.1 ([10], generalized curvature dimension inequality) *We say that L satisfies the* generalized curvature dimension inequality $\mathrm{CD}(\rho_1,\rho_2,\kappa,d)$ *on $\mathbb{M}$ if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $0 < d < \infty$ such that the inequality*

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f)$$

holds for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$.

The inequality $CD(\rho_1,\rho_2,\kappa,d)$ turns out to be equivalent to lower bounds on intrinsic curvature tensors in [10]. The following is an exemplary curvature condition implying $CD(\rho_1,\rho_2,\kappa,d)$ on CR manifold.

With the curvature inequality condition assumed, various aspects on sub-Riemannian manifolds have been discovered in [10], [7], [8], [9], [11], [15], [12]. In particular, we have the following essential properties - two-sided heat kernel bounds and volume doubling property of balls with exponential term.

Proposition 3.2.1 *(Remark 3, Theorem 13, Theorem 12 in [9]) If we assume the curvature condition $CD(-K,\rho_2,\kappa,d)$, $K > 0$ on $\mathbb{M}$, for any $x,y \in \mathbb{M}$, $t > 0$, $r > 0$,*

$$p_t(x,y) \geq \frac{C_1}{\mu(B(x,\sqrt{t}))}\exp\left(-\frac{D}{2d}\frac{d(x,y)^2}{t} - C_2K(t + d(x,y)^2)\right) \tag{3.8}$$

$$p_t(x,y) \leq \frac{C_3}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}}\exp\left(C_4Kt - \frac{d(x,y)^2}{5t}\right) \tag{3.9}$$

$$\mu(B(x,2r)) \leq C_{d1}\exp(C_{d2}Kr^2)\mu(B(x,r)), \tag{3.10}$$

where $D = d\left(1 + \frac{3\kappa}{2\rho_2}\right)$, $C_1, C_2, C_3, C_4, C_{d1}, C_{d2}$ are positive and determined by ρ_2, κ, d.

Denote $Q = \log_2 C_{d1}$. (3.10) implies that for any $\lambda > 1$,

$$\begin{aligned}\frac{\mu(B(x,\lambda r))}{\mu(B(x,r))} &\leq C_{d1}^{\lceil \log_2 \lambda \rceil} \exp(C_{d2} K \sum_{i=0}^{\lceil \log_2 \lambda \rceil - 1} (2^i r)^2) \\ &\leq C_{d1} \lambda^Q \exp(\frac{4C_{d2}}{3} K (\lambda r)^2).\end{aligned} \tag{3.11}$$

This doubling property allows us to estimate $\mu(B(x,\sqrt{t}))$ by $\mu(B(y,\sqrt{t}))$:

$$\begin{aligned}\mu(B(x,\sqrt{t})) &\leq \mu(B(y,\sqrt{t}+d(x,y))) \\ &\leq \mu(B(y,\sqrt{t}))2C_{d1}\left(1+\frac{d(x,y)^2}{t}\right)^{Q/2} \exp\left(\frac{8C_{d2}}{3}K(t+d(x,y)^2)\right).\end{aligned}$$

So we modify the upper bound of heat kernel (3.9) with the volume of a single ball, i.e., for $C_5, C_6 > 0$ depending on ρ_2, κ, d,

$$p_t(x,y) \leq \frac{C_5}{\mu(B(x,\sqrt{t}))} \exp\left(C_6 K(t+d(x,y)^2) - \frac{d(x,y)^2}{6t}\right). \tag{3.12}$$

Note that $1 + A \leq C(\epsilon)e^{\epsilon A}$ for $\forall \epsilon > 0, A \geq 0$ is applied.

Remark 3.2.1 *As mentioned in [9], the square in the exponent of volume doubling property might not be optimal. For instance, in the Riemannian manifold with Ricci tensor bounded below by* $-K < 0$*, by the Bishop-Gromov comparison theorem we have* $V(x,\lambda r) \leq V(x,r)\lambda^n \exp(\sqrt{(n-1)K}(\lambda r))$ *where* $\lambda > 1$ *and* $V(x,r)$ *is the Riemannian measure of the ball* $B(x,r)$*.*

This yields the difference of the exponent in (3.3) and (3.4).

Remark 3.2.2 *([7]) Notice that the positive solution* u *carries an additional condition in the Li-Yau type inequality, Corollary 3.1.4. Due to the technical reason in the proof of Theorem 6.1 in [10],* u *needs to be contained in* $\mathcal{A}_\epsilon = \{f \in C_b^\infty(\mathbb{M}) : f - \epsilon \geq 0, \sqrt{\Gamma(f-\epsilon)}, \sqrt{\Gamma^Z(f-\epsilon)} \in L^2(\mathbb{M})\}$*. Same restriction is required for the log-Sobolev inequality in [7].*

3.3 Poincaré inequality on the ball

3.3.1 Lower bound of the Dirichlet heat kernel on the ball

Throughout this section, L satisfies $CD(-K, \rho_2, \kappa, d)$, $K > 0$ on $\mathbb{M}$.

To adapt Kusuoka and Stroock's idea [44], the necessary ingredients will be two-sided heat kernel bound (3.8),(3.9) and doubling (3.11).

Denote $B = B(x_0, r)$, sub-Riemannian ball centered at x_0 with radius r. On the ball B, the Dirichlet heat kernel $p_t^{B,D}(x, y)$ will be defined by the transition probability

$$p_t^{B,D}(x, y)d\mu(y) = P[\zeta > t, X(t) \in d\mu(y)],$$

where $X(t)$ is the associated Markov process of the semigroup operator $P_t = e^{tL}$ with $X(0) = x$, and the lifetime of X in B is $\zeta = \inf\{t > 0, X(t) \notin B\}$.

First, the lower bound of the Dirichlet heat kernel for close x, y can be obtained by the argument of Kusuoka and Stroock [44]:

Lemma 3.3.1 *For any $k \in (0, 1)$, there exists $C_\alpha = C(k, \rho_2, \kappa, d) \in (1, \infty)$ such that for any $x_0 \in \mathbb{M}$, $r > 0$ and $\alpha = \sqrt{\frac{1}{C_\alpha(Kr^2+1)}} \in (0, 1)$, the Dirichlet heat kernel on $B = B(x_0, r)$ has lower bound*

$$p_t^{B,D}(x, y) \geq \frac{c}{\mu(B(x, \sqrt{t}))} \exp\left(-C\frac{d(x, y)^2}{t}\right) \tag{3.13}$$

for all $t \in (0, (\alpha r)^2]$ and $x, y \in B(x_0, kr)$ such that $d(x, y) \leq \alpha r$.
Here $c, C > 0$ depend only on ρ_2, κ, d.

Proof Let $\alpha = (C_\alpha(Kr^2 + 1))^{-\frac{1}{2}} \in (0, 1)$ with some $C_\alpha > 1$ which will be determined later. Note that $K(\alpha r)^2 \leq C_\alpha^{-1} \leq 1$.

Let $d(x, y) \leq \alpha r$ and $t \leq (\alpha r)^2$. The Dirichlet heat kernel can be written by the heat kernel of $\mathbb{M}$ and the lifetime of the process in the domain. That is,

$$p_t^{B,D}(x, y) = p_t(x, y) - \mathbb{E}^x[p_{t-\zeta}(X(\zeta), y), \zeta < t], \quad \zeta = \inf\{t > 0, X(t) \notin B(x_0, r)\}.$$

The lower bound (3.8) on the heat kernel $p_t(x, y)$ over the whole manifold yields

$$p_t(x, y) \geq \frac{C_1 e^{-2C_2}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{D}{2d}\frac{d(x, y)^2}{t}\right).$$

If we use upper bound (3.9) on the heat kernel in the expectation, we have

$$p_{t-\zeta}(X(\zeta), y) \leq \frac{C_3 \exp\left(C_4 K(t - \zeta) - \frac{d(X(\zeta),y)^2}{5(t-\zeta)}\right)}{\mu(B(X(\zeta), \sqrt{t - \zeta}))^{1/2}\mu(B(y, \sqrt{t - \zeta}))^{1/2}}.$$

The balls can be replaced by concentric balls using the doubling property (3.11) as follows.

$$\frac{\mu(B(x,\sqrt{t}))}{\mu(B(X(\zeta),\sqrt{t-\zeta}))} \leq \frac{\mu(B(X(\zeta),3r))}{\mu(B(X(\zeta),\sqrt{t-\zeta}))} \leq C_{d1}\left(\frac{3r}{\sqrt{t-\zeta}}\right)^{Q} \exp(\frac{4C_{d2}K}{3}(3r)^2),$$
$$\frac{\mu(B(x,\sqrt{t}))}{\mu(B(y,\sqrt{t-\zeta}))} \leq \frac{\mu(B(y,2\alpha r))}{\mu(B(y,\sqrt{t-\zeta}))} \leq C_{d1}\left(\frac{2\alpha r}{\sqrt{t-\zeta}}\right)^{Q} \exp(\frac{4C_{d2}K}{3}(2\alpha r)^2).$$

With $d(X(\zeta),y) \geq r(1-k)$ and $t-\zeta \leq t \leq (\alpha r)^2$, the above controls imply that

$$\begin{aligned}&\mathbb{E}^x[p_{t-\zeta}(X(\zeta),y),\zeta<t]\\ &\quad\leq \frac{C_3 C_{d1} e^{3C_{d2}+C_4}}{\mu(B(x,\sqrt{t}))}\exp\left(-\frac{r^2(1-k)^2}{10t}+6C_{d2}Kr^2\right)\cdot C_\nu \exp\left(-\nu\frac{1}{\alpha^2}\right).\end{aligned}$$

The last term came from $\mathbb{E}^x\left[\left(\frac{\sqrt{(3r)(2\alpha r)}}{\sqrt{t-\zeta}}\right)^{Q}\exp\left(-\frac{r^2(1-k)^2}{10(t-\zeta)}\right)\right] \leq C_\nu \exp\left(-\nu\frac{1}{\alpha^2}\right)$, which holds if we choose $C_\nu \geq \left(\frac{60Q}{e(1-k)^2}\right)^{Q/2}$, $\nu \leq \frac{(1-k)^2}{20}$.

Combining these upper and lower estimates with $t \leq (\alpha r)^2$, $d(x,y) \leq \alpha r$,

$$\begin{aligned}&p_t^{B,D}(x,y)\\ &\geq \frac{C_1 e^{-2C_2}}{\mu(B(x,\sqrt{t}))} e^{\left(-\frac{D}{2d}\frac{d(x,y)^2}{t}\right)}\left[1-C\exp\left(-\frac{r^2}{t}(\frac{(1-k)^2}{10}-\frac{D}{2d}\alpha^2)+6C_{d2}Kr^2-\frac{\nu}{\alpha^2}\right)\right],\end{aligned}$$

where $C=C_3C_{d1}C_\nu C_1^{-1}e^{3C_{d2}+C_4+2C_2}$.

Choose α small enough for $\left[1-C\exp\left(-\frac{r^2}{t}(\frac{(1-k)^2}{10}-\frac{D}{2d}\alpha^2)+6C_{d2}Kr^2-\frac{\nu}{\alpha^2}\right)\right] \geq \frac{1}{2}$.

Then we conclude

$$p_t^{B,D}(x,y) \geq \frac{c}{\mu(B(x,\sqrt{t}))}\exp\left(-C\frac{d(x,y)^2}{t}\right),$$

for all $t \leq (\alpha r)^2$, $d(x,y) \leq \alpha r$, where $c = 2^{-1}C_1 e^{-2C_2}, C=\frac{D}{2d} > 0$ depend on ρ_2, κ, d.

For instance, if we pick large $C_\alpha = C(\rho_2,\kappa,d,k) > 0$ such as

$$C_\alpha \geq \max\left\{\frac{\frac{D}{2d}+\ln(2C_3C_{d1}C_\nu C_1^{-1}e^{3C_{d2}+C_4+2C_2})+6C_{d2}}{\nu+\frac{(1-k)^2}{10}}, \left(\frac{2d}{D}\frac{(1-k)^2}{10}\right)^{-1}\right\},$$

then our choice of

$$\alpha^2 = \frac{1}{C_\alpha(Kr^2+1)}, \tag{3.14}$$

satisfies the estimates above.

■

Next step is the lower bound of Dirichlet heat kernel for any x, y in the smaller ball which is followed by the chain argument. Note that our lemma holds for any $r > 0$ with the exponential square of radius, while the classic lemma holds only for $0 < r \leq 1$.

Lemma 3.3.2 *For any $0 < k < 1$ and $0 < \delta < 1$, there exists $0 < c < 1$, $C > 0$ such that for any $x_0 \in \mathbb{M}$ and $r > 0$, the Dirichlet heat kernel on the ball $B = B(x_0, r)$ has lower bound*

$$p_t^{B,D}(x,y) \geq \frac{c\exp(-CKr^2)}{\mu(B(x_0,kr))}$$

for all $x, y \in B(x_0, kr)$ and $\delta r^2 \leq t \leq r^2$.

Proof Choosing $\alpha \in (0,1)$ of (3.14) in the previous lemma, for all $t \leq (\alpha r)^2$, $x, y \in B(x_0, kr)$, $d(x,y) \leq \alpha r$,

$$p_t^{B,D}(x,y) \geq \frac{c}{\mu(B(x,\sqrt{t}))}\exp(-C\frac{d(x,y)^2}{t}).$$

Now let x, y be any points in $B(x_0, kr)$ and $\delta r^2 \leq t \leq r^2$. Set $n = \lceil 16\alpha^{-2} \rceil$, then $16\alpha^{-2} \leq n \leq 17\alpha^{-2}$.

We choose $\{\xi_i\}_{i=0,1,\cdots,2n} \subset B(x_0, kr)$ such that

$$\xi_0 = x, \quad \xi_n = x_0, \quad \xi_{2n} = y,$$
$$d(\xi_k, \xi_{k+1}) \leq \frac{r}{n} \leq \frac{\alpha r}{4}.$$

Let $\tau = \frac{t}{2n}$. Since $\sqrt{\tau} \leq \frac{\alpha r}{4}$, if $\eta_k \in B(\xi_k, \sqrt{\tau})$, then $d(\eta_k, \eta_{k+1}) \leq \alpha r$.

By the previous lemma,

$$
\begin{aligned}
p_\tau^{B,D}(\eta_k,\eta_{k+1}) \geq & \frac{c}{\mu(B(\eta_k,\sqrt{\tau}))}\exp(-C\frac{d(\eta_k,\eta_{k+1})^2}{\tau}) \\
\geq & \frac{cC_{d1}^{-1}\exp(-C_{d2}K(\alpha r)^2)}{\mu(B(\xi_k,\sqrt{\tau}))}\exp(-C\frac{d(\eta_k,\eta_{k+1})^2}{\tau}).
\end{aligned}
$$

And we see that

$$
\frac{d(\eta_k,\eta_{k+1})^2}{\tau} \leq \left(\frac{d(\xi_k,\xi_{k+1})}{\sqrt{\tau}}+2\right)^2 \leq \left(\frac{r\sqrt{2}}{\sqrt{nt}}+2\right)^2 \leq \frac{4}{\delta n}+8.
$$

Observing

$$
\begin{aligned}
p_t^{B,D}(x,y) \geq \int_{B(\xi_{2n-1},\sqrt{\tau})} \cdots \int_{B(\xi_1,\sqrt{\tau})} & p_\tau^{B,D}(x,\eta_1) \\
& \cdot p_\tau^{B,D}(\eta_1,\eta_2)\cdots p_\tau^{B,D}(\eta_{2n-1},y)d\eta_1\cdots d\eta_{2n-1},
\end{aligned}
$$

we obtain

$$
p_t^{B,D}(x,y) \geq \frac{1}{\mu(B(x_0,\sqrt{\tau}))}\left(cC_{d1}^{-1}\exp(-C(\frac{4}{\delta n}+8)-C_{d2}K(\alpha r)^2)\right)^{2n}.
$$

Doubling property (3.11) yields

$$
\frac{1}{\mu(B(x_0,\sqrt{\tau}))} \geq \frac{1}{\mu(B(x_0,r))} \geq \frac{C_{d1}^{-1}k^Q\exp(-\frac{4C_{d2}}{3}Kr^2)}{\mu(B(x_0,kr))}.
$$

Also since $cC_{d1}^{-1}\exp(-8C)<1$ and $n\leq 17\alpha^{-2}=17C_\alpha(Kr^2+1)$ from (3.14),

$$
\begin{aligned}
& \left(cC_{d1}^{-1}\exp\left(-C(\frac{4}{\delta n}+8)-C_{d2}K(\alpha r)^2\right)\right)^{2n} \\
& \geq \exp\left(-\frac{8C}{\delta}-(8C-\ln(cC_{d1}^{-1})+C_{d2})\cdot 34C_\alpha(Kr^2+1)\right).
\end{aligned}
$$

This concludes our lemma

$$
p_t^{B,D}(x,y) \geq \frac{c'\exp(-C'Kr^2)}{\mu(B(x_0,kr))},
$$

where

$$
\begin{aligned}
c' &= C_{d1}^{-1}k^Q\exp\left(-\frac{8C}{\delta}-(8C-\ln(cC_{d1}^{-1})+C_{d2})(34C_\alpha)\right), \\
C' &= \frac{4C_{d2}}{3}+(8C-\ln(cC_{d1}^{-1})+C_{d2})(34C_\alpha)
\end{aligned}
$$

are determined by ρ_2,κ,d,k,δ. ■

3.3.2 Proof of Theorem 3.1.1

In this section, we utilize Dirichlet, Neumann heat semigroup which can be found in [63], [58], [44], then we will follow the arguments in [44] to prove Poincaré inequality (3.4).

Let $B = B(x_0, r)$. Define a subspace $D^\infty \subset C^\infty(B)$ as a collection of functions f satisfying $-\int_B gLf d\mu = \int_B \Gamma(g, f) d\mu$ for $\forall g \in C^\infty(B)$. Note that $C_0^\infty(B) \subset D^\infty \subset C^\infty(B)$.
The Dirichlet form $\mathcal{E}(f, g) = \int_B \Gamma(f, g) d\mu$ on D^∞ is closable in $L^2(B)$, and by closing it we gain a Dirichlet form and associated Markov heat semigroup $P_t^{B,N}$ with Neumann boundary condition.

If we denote $p_t^{B,N}$ by the Neumann heat kernel over B, it will be a smooth kernel of the Neumann heat semigroup and its associated transition probability function. Naturally, since $C_0^\infty(B) \subset D^\infty$, the Neumann heat kernel dominates the Dirichlet heat kernel, i.e., $p_t^{B,N} \geq p_t^{B,D}$.

Proof [Proof of Theorem 3.1.1 .] We will prove the inequality with $B(x_0, r/2)$ on the left hand side. Then by the Whitney type covering lemma (section 5 in [39]), we can match the balls on the both sides. The Whitney decomposition only requires a doubling property in the domain of argument - for example, in $B(x_0, 10r)$, the doubling property holds with fixed constant $C_{d1} \exp(C_{d2} K (10r)^2)$, which will be multiplied at the end following the argument. (See Cor.5.3.5, Thm 5.6.5, Thm 5.6.6 in [58], see Thm 4.2 and Thm 4.3 in [8])

From the previous lemma, for $x, y \in B(x_0, r/2)$,

$$p_{r^2}^{B(x_0,r),N}(x, y) \geq \frac{ce^{-CKr^2}}{\mu(B(x_0, r/2))}.$$

For any $f \in C^\infty(B)$ and $x \in B(x_0, r/2)$,

$$\begin{aligned} P_{r^2}^{B(x_0,r),N}&(f - P_{r^2}^{B(x_0,r),N} f(x))^2(x) \\ &\geq \frac{ce^{-CKr^2}}{\mu(B(x_0,r/2))} \int_{B(x_0,r/2)} (f(y) - P_{r^2}^{B(x_0,r),N} f(x))^2 d\mu(y) \\ &\geq \frac{ce^{-CKr^2}}{\mu(B(x_0,r/2))} \int_{B(x_0,r/2)} (f(y) - f_{B(x_0,r/2)})^2 d\mu(y). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{B(x_0,r)} P_{r^2}^{B(x_0,r),N}(f - P_{r^2}^{B(x_0,r),N} f(x))^2(x) d\mu(x) &\leq \int_{B(x_0,r)} (f^2 - P_{r^2}^{B(x_0,r),N} f(x)^2) d\mu(x) \\ &= \int_0^{r^2} \int_{B(x_0,r)} -\frac{d}{dt}(P_t^{B(x_0,r),N} f(x))^2 d\mu(x) dt \\ &= \int_0^{r^2} \int_{B(x_0,r)} -2P_t^{B(x_0,r),N} f(x) L P_t^{B(x_0,r),N} f(x) d\mu(x) dt \\ &= \int_0^{r^2} \int_{B(x_0,r)} 2\Gamma(P_t^{B(x_0,r),N} f(x)) d\mu(x) dt \\ &\leq 2r^2 \int_{B(x_0,r)} \Gamma(f) d\mu, \end{aligned}$$

where the last inequality comes from $\frac{d}{dt}\Gamma(P_t f) \leq 0$. And we obtain our desired conclusion

$$\int_{B(x_0,r/2)} (f(x) - f_{B(x_0,r/2)})^2 d\mu(x) \leq C'_{p1} r^2 e^{C'_{p2}Kr^2} \int_{B(x_0,r)} \Gamma(f) d\mu,$$

with $C'_{p1} = 2/c, C'_{p2} = C$. ■

3.3.3 Sobolev inequality and L^p mean value estimate

This section is dedicated to the Sobolev inequality 3.2 and L^p mean value inequality for subharmonic functions. These are essential to prove Theorem 3.1.2 and 3.1.3. Throughout this section, harmonic (resp.subharmonic) functions are $f \in \mathrm{Dom}(L)$ satisfying $Lf = 0$ (resp.$Lf \geq 0$). As mentioned in the introduction, even if the scale-invariant Sobolev inequality is a consequence of the Poincaré inequality (theorem 2.2 in [59]), we can provide a direct proof from the heat kernel bounds of [9]. (section.10 in [60], and the last section of [67]).

Proposition 3.3.1 ([59], [60], Sobolev inequality on balls) *Assume* $CD(-K, \rho_2, \kappa, d)$, $K > 0$ *on* $\mathbb{M}$. *Then for any* $x \in \mathbb{M}$, $0 < r$, $B = B(x, r)$, $f \in C_0^\infty(B(x, r))$,

$$\left(\frac{1}{\mu(B)} \int_B |f|^{\frac{2Q}{Q-2}} d\mu\right)^{\frac{Q-2}{2Q}} \leq C_1 r e^{C_e K r^2} \left(\frac{1}{\mu(B)} \int_B (\Gamma(f) + r^{-2}|f|^2) d\mu\right)^{\frac{1}{2}}, \qquad (3.15)$$

where $Q = \log_2 C_{d1}$ *in (3.10),* $C_1, C_e > 0$ *depend only on* ρ_2, κ, d.

Proof By the upper bound of the heat kernel (3.12)

$$p_t^{B,D}(x, y) \leq p_t(x, y) \leq \frac{C_5}{\mu(B(x, \sqrt{t}))} \exp\left(C_6 K (t + d(x, y)^2) - \frac{d(x, y)^2}{6t}\right),$$

where $B = B(x_0, r)$. Since $0 < t \leq r^2$ and $d(x, y) \leq 2r$ for $x, y \in B$, by (3.11)

$$\mu(B(x_0, r)) \leq \mu(B(x, 2r)) \leq C \left(\frac{r}{\sqrt{t}}\right)^Q \exp(cKr^2) \mu(B(x, \sqrt{t})).$$

Therefore, the Dirichlet heat kernel will be bounded from above by

$$p_t^{B,D}(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{cKr^2} \leq \frac{C'}{\mu(B(x_0, r))} r^Q t^{-Q/2} e^{c'Kr^2}.$$

Proposition 10.1 in [60] (also [67]) states that

$$\begin{aligned} &\|P_t^{B,D}\|_{1\to\infty} \leq C_0 t^{-Q/2}, \quad \forall 0 < t < t_0 \\ &\quad \Longrightarrow \|f\|_{\frac{2Q}{Q-2}} \leq C_0^{1/Q} \left(C \|\sqrt{\Gamma(f)}\|_2 + t_0^{-1/2} \|f\|_2\right), \quad \forall f \in C_0^\infty(B). \end{aligned}$$

Taking $C_0 = \frac{C'}{\mu(B(x_0, r))} r^Q e^{c'Kr^2}$ and $t_0 = r^2$ with $A + B \leq (2A^2 + 2B^2)^{1/2}$, the Sobolev inequality (3.15) is proved. ■

Once the scale-invariant Sobolev embedding is acquired, our goal of this section, L^p mean value estimate, can be obtained through the Moser's iteration. One can find the arguments for the Riemannian case in [58].

Lemma 3.3.3 (One step of Moser's iteration) *We assume that* $\mathbb{M}$ *satisfies* $CD(-K, \rho_2, \kappa, d)$, $K > 0$. *For any subharmonic function* $u(x) \geq 0$, *i.e.* $Lu(x) \geq 0$, *and* $0 < R_1 < R_2 \leq R$, $p \geq 2$,

$$\int_{B(R_1)} u^{p\theta} d\mu \leq C_2 e^{2C_e K R^2} \frac{R^2}{(R_2 - R_1)^2} V^{1-\theta} \left(\int_{B(R_2)} u^p d\mu \right)^{\theta}, \tag{3.16}$$

where $\theta = 1 + \frac{2}{Q}$, $B(\cdot) = B(x_0, \cdot)$, $V = \mu(B(R))$.

The proof of the lemma is exactly the same argument to the standard case [58] when we have the cutoff functions as follows.

Remark 3.3.4 *For any* $0 < R_1 < R_2 < \infty$, *there exists a Lipschitz continuous cut-off function* $\psi \geq 0$ *satisfying* $\psi|_{B(R_1)} = 1$, $\operatorname{supp} \psi \subset B(R_2)$, $\sqrt{\Gamma(\psi)} \leq \frac{C}{R_2 - R_1}$ *almost everywhere for some* $C > 0$ *which is independent to* R_1, R_2. *See theorem 1.5 in [30], lemma 3.6 in [21] and [62].*

Again, by the standard iteration of the above lemma (section 2.2 in [58]), we prove L^p mean value estimate.

Theorem 3.3.5 (L^p mean value inequality, $p \geq 2$) *For any* $0 < \delta < 1$, *any* $p \geq 2$, *and any non-negative subsolution* u *of* $Lu = 0$ *in a ball* $B(R)$ *of volume* V,

$$\sup_{\delta B}\{u^p\} \leq C_3 e^{QC_e K R^2} (1 - \delta)^{-Q} \left(V^{-1} \int_B u^p d\mu \right). \tag{3.17}$$

Corollary 3.3.6 (L^p mean value inequality, $0 < p < 2$) *L^p mean value inequality (3.17) also holds for any* $0 < p < 2$ *with the constant* C_3 *replaced by some* $C_4 = C(Q, p)$. *In particular, for* $p = 1$

$$\sup_{\delta B}\{u\} \leq C_m e^{c_m K R^2} (1 - \delta)^{-Q} \left(\frac{1}{\mu(B)} \int_B u d\mu \right),$$

where $C_m, c_m > 0$ *depend only on* ρ_2, κ, d.

3.4 Uniqueness of the positive solution

3.4.1 Minimality of the heat semigroup for positive solutions

To prove Theorem 3.1.2, we reduce the question to the zero initial data. Following Lemma 3.4.1 enables the reduction. This section is based on the idea of [23].

Lemma 3.4.1 (minimality of the heat semigroup) *Let $u \in C(\mathbb{M} \times (0,T))$ be a non-negative supersolution of the heat equation (3.5) with initial data $f \in L^2_{loc}(\mathbb{M})$, $f \geq 0$.*
Then $P_t f(x) = \int_{\mathbb{M}} p_t(x,y) f(y) d\mu(y)$ is a smooth solution of (3.5) satisfying $P_t f \xrightarrow{L^2_{loc}} f$ as $t \to 0$ and $u(\cdot, t) \geq P_t f$.

Proof For any $\Omega \Subset \mathbb{M}$, we denote $P_t^{\Omega,D}$ the Dirichlet heat semigroup associated with L on Ω. Using the maximum principle for $P_t^{\Omega,D} f - u(\cdot,t)$, we have

$$u(x,t) \geq P_t^{\Omega,D} f(x), \quad \forall x \in \Omega$$

Denote $f_k = f 1_{\Omega_k} \in L^2(\mathbb{M})$ for the exhaustion $\{\Omega_k\}$. As shown above, $u(\cdot,t) \geq P_t^{\Omega_k,D} f \geq P_t^{\Omega_k,D} f_i$ for all i. Since $P_t^{\Omega_k,D} f_i \xrightarrow{L^2(\mathbb{M})} P_t f_i$ as $k \to \infty$, we have $u(\cdot,t) \geq P_t f_i$ almost everywhere for all i. Therefore,

$$u(\cdot,t) \geq P_t f \quad \text{almost everywhere.}$$

To prove that the smooth $P_t f$ solves the heat equation, first we see that $P_t f \in L^1_{loc}(\mathbb{M})$ from the above estimate. Denote

$$u_k = P_t(\min(f,k) 1_{\Omega_k}),$$

then u_k is a smooth solution of the subelliptic heat equation and $u_k \nearrow P_t f$ as $k \to \infty$ at any $(x,t) \in \mathbb{M} \times (0,T)$.

For any $\varphi \in C_0^\infty(\mathbb{M} \times (0,T))$, since $P_t f \in L^1_{loc}(\mathbb{M})$,

$$|(\partial_t \varphi + L\varphi) u_k| \leq (\sup |\partial_t \varphi + L\varphi|) 1_{\operatorname{supp} \varphi} P_t f \in L^1(\mathbb{M}), \quad \forall k \in \mathbb{N}.$$

This allows us to take the limit of the integrand on the left hand side of

$$\int_0^T \int_{\mathbb{M}} (\partial_t \varphi + L\varphi) u_k d\mu dt = \int_0^T \int_{\mathbb{M}} \varphi \left(L - \partial_t\right) u_k d\mu dt = 0.$$

Therefore $P_t f$ is a distributional solution of the subelliptic heat equation, and also it is smooth by the smooth convergence of u_k to $P_t f$ and the hypoellipticity of $L - \partial_t$.

Once the smoothness of $P_t f$ and $u \geq P_t f$ are proved, the initial condition is straightforward as follows : On any $\Omega \Subset \mathbb{M}$,

$$P_t(f 1_\Omega) \leq P_t f \leq u(\cdot, t).$$

When $t \to 0$, $u \xrightarrow{L^2(\Omega)} f$ and $P_t(f1_\Omega) \xrightarrow{L^2(\mathbb{M})} f1_\Omega$. Hence $P_t f \xrightarrow{L^2(\Omega)} f$. ■

3.4.2 Proof of Theorem 3.1.2

From the minimality Lemma 3.4.1, for any non-negative continuous solution u of (3.5),

$$w(x,t) = u(x,t) - P_t u(x,0)$$

is a non-negative solution of (3.5) with zero initial data. Thus we can reduce the uniqueness of the positive solution to the zero initial data case.

Let $w(x,t)$ be any non-negative solution of the heat equation (3.5) with initial data $f \equiv 0$.

Define $v(x,t) = \int_0^t w(x,s)ds$. Our goal is to show $v \equiv 0$, and so is w.

Remark 3.4.2 *$v(x,t) = \int_0^t w(x,s)ds$ is a non-negative solution of the heat equation (3.5) with zero initial data, and subharmonic in x, i.e. $Lv(\cdot,t) = \int_0^t Lw(\cdot,s)ds = w(\cdot,t) \geq 0$.*

The following growth estimate condition is originally suggested by Tikhonov for the uniqueness of the solution for the heat equation.

Proposition 3.4.1 (Growth estimate of the solution, Tikhonov's condition) *For any $\epsilon > 0$ and $0 \leq t \leq \epsilon$, if $v \in C(\mathbb{M} \times (0, \epsilon))$ is a non-negative solution of the subelliptic heat equation (3.5) satisfying $Lv(\cdot, t) \geq 0$, then*

$$v(x, t) \leq C_1 \exp(C_2 d^2(p, x)),$$

where $C_1 = C_1(\epsilon) > 0$, $C_2 = C_2(\epsilon) > 0$, and $d(p, \cdot)$ is the distance from a fixed $p \in \mathbb{M}$.

Proof Let $B = B(x, d(p, x) + 1)$. Fix $T > 0$. From the minimality Lemma 3.4.1,

$$v(p, t + T) \geq P_T v(\cdot, t) = \int_M p_T(p, y) v(y, t) d\mu(y) \geq \int_B p_T(p, y) v(y, t) d\mu(y).$$

From the curvature condition $CD(-K, \rho_2, \kappa, d)$, the lower bound of heat kernel (3.8) is

$$p_T(p, y) \geq C_3 \exp(-C_4 d^2(p, y)),$$

where $C_3 = C_3(p, T, K, \rho_2, \kappa, d) > 0$, $C_4 = C_4(T, K, \rho_2, \kappa, d) > 0$.

By the triangle inequality $d(p, y) \leq 2d(p, x) + 1$ for $y \in B$,

$$\int_B v(y, t) d\mu(y) \leq C_5 \exp(C_6 d^2(p, x)) \; v(p, t + T).$$

By L^1 mean value estimate of Corollary 3.3.6 for the subharmonic function $v(\cdot, t)$

$$v(x, t) \leq C_7 \exp(C_8 d(p, x)^2) \int_B v(y, t) d\mu(y),$$

where $C_7, C_8 > 0$ depend on K, ρ_2, κ, d. Therefore, we obtain

$$v(x, t) \leq C_9 \exp(C_{10} d^2(p, x)) v(p, t + T),$$

where the constants depend on $p, T, K, \rho_2, \kappa, d$. As t varies from 0 to ϵ, $v(p, t + T)$ remains uniformly bounded in t. So we have the desired conclusion. ■

Together with the previous proposition, the proof of Theorem 3.1.2 is finished by the following proposition.

Proposition 3.4.2 *If $v(x,t)$ is a solution of (3.5) with initial $f(x) \equiv 0$ satisfying*

$$|v(x,t)| \leq C_1 \exp C_2 d^2(p,x)$$

for some positive C_1, C_2, then $v \equiv 0$.

Existence of Lipschitz cut-off function and integration by part allow us to follow exactly the same proof of corollary 11.10 in [32].

3.5 Uniqueness of L^p solution

3.5.1 Proof of Theorem 3.1.3, $p > 1$

For $p = \infty$, the uniqueness of the L^∞ solution, or equivalently the stochastic completeness of $\mathbb{M}$, can be found in [10]. If $p \in (1, \infty)$, without any curvature assumption, the uniqueness follows immediately by adapting the idea of [51].

Theorem 3.5.1 *Let $v(x,t)$ is a non-negative function defined on $\mathbb{M} \times (0,T)$ with*

$$\left(\frac{\partial}{\partial t} - L\right) v(x,t) \leq 0$$

$$v \xrightarrow{L^p_{loc}} 0 \text{ as } t \to 0$$

$$v(\cdot, t) \in L^p(\mathbb{M}) \quad \forall t \in (0,T),$$

then $v(x,t) \equiv 0$ on $\mathbb{M} \times (0,T)$.
In particular, any L^p solution of the heat equation is uniquely determined by its initial data in $L^p(\mathbb{M})$.

Proof Fix $x_0 \in \mathbb{M}$ an arbitrary base point.

From remark 3.3.4, we choose $\psi(x) \in C_0(B(x_0, 2R))$ a cut-off function satisfying $\psi|_{B(x_0,R)} \equiv 1$, $0 \leq \psi \leq 1$, $\|\sqrt{\Gamma(\psi)}\|_\infty \leq \frac{C}{R}$ for some $C > 0$.

Since v is a subsolution with the zero initial data, for any $\tau \in (0,T)$,

$$\int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x,t) Lv(x,t) d\mu(x) dt \geq \int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1} \frac{\partial v}{\partial t} d\mu(x) dt$$
$$= \frac{1}{p} \int_0^\tau \frac{\partial}{\partial t} \left(\int_{\mathbb{M}} \psi^2(x) v^p d\mu(x) \right) dt = \frac{1}{p} \int_{\mathbb{M}} \psi^2(x) v^p(x,\tau) d\mu(x).$$

On the other hand, integrating by parts yields

$$\int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x,t) Lv(x,t) d\mu(x) dt$$
$$= -\int_0^\tau \int_{\mathbb{M}} 2\psi v^{p-1} \Gamma(\psi, v) d\mu dt - \int_0^\tau \int_{\mathbb{M}} \psi^2 (p-1) v^{p-2} \Gamma(v) d\mu dt.$$

On the right hand side, observing

$$0 \le \left(\sqrt{\frac{2}{p-1}\Gamma(\psi)} v - \sqrt{\frac{p-1}{2}\Gamma(v)} \psi \right)^2 \le \frac{2}{p-1}\Gamma(\psi) v^2 + 2\Gamma(\psi, v)\psi v + \frac{p-1}{2}\Gamma(v)\psi^2,$$

we obtain the following estimate.

$$\int_0^\tau \int_{\mathbb{M}} \psi^2(x) v^{p-1}(x,t) Lv(x,t) d\mu(x) dt$$
$$\le \int_0^\tau \int_{\mathbb{M}} \frac{2}{p-1}\Gamma(\psi) v^p d\mu dt - \int_0^\tau \int_{\mathbb{M}} \frac{p-1}{2} \psi^2 v^{p-2} \Gamma(v) d\mu dt$$
$$= \int_0^\tau \int_{\mathbb{M}} \frac{2}{p-1}\Gamma(\psi) v^p d\mu dt - \frac{2(p-1)}{p^2} \int_0^\tau \int_{\mathbb{M}} \psi^2 \Gamma(v^{p/2}) d\mu dt.$$

Combining with the previous conclusion and the assumption $|\sqrt{\Gamma(\psi)}| \le \frac{C}{R}$,

$$\int_{\mathbb{M}} \psi^2(x) v^p(x,\tau) d\mu(x) + \frac{2(p-1)}{p} \int_0^\tau \int_{\mathbb{M}} \psi^2 \Gamma(v^{p/2}) d\mu dt \le \frac{2pC^2}{(p-1)R^2} \int_0^\tau \int_{\mathbb{M}} v^p d\mu dt.$$

As $R \to \infty$, since $\Gamma(v^{p/2}) \ge 0$, we have

$$\int_{\mathbb{M}} v^p(x,\tau) d\mu(x) = 0 \quad \forall \tau \in (0,T).$$

Thus, $v \equiv 0$. ∎

3.5.2 Hamilton's inequality

Before we move on to L^1 solutions, we will prove the gradient estimate of the logarithm of the heat kernel. We will apply subelliptic version of Hamilton's inequality which was originally proved for closed Riemannian manifolds in [33], then for non-compact Riemannian manifolds in [43].

Proposition 3.5.1 (Hamilton's inequality) *Assume that $\mathbb{M}$ satisfies the curvature condition $CD(-K, \rho_2, \kappa, d)$. If a positive solution $u \in \mathcal{A}_\epsilon$ to the subelliptic heat equation satisfies $u \leq M$ on $\mathbb{M} \times (0, T)$ for some $M > 0$ and $0 < T \leq \infty$, one has*

$$t\Gamma(\ln u(x,t)) \leq \left(1 + \frac{2\kappa}{\rho_2} + 2Kt\right) \ln\left(\frac{M}{u(x,t)}\right) \tag{3.18}$$

for all $(x, t) \in \mathbb{M} \times (0, T)$.

Proof By Theorem 3.1.2, it suffices to show that the estimate holds for $u = P_t f \in \mathcal{A}_\epsilon > 0$. (See Remark 3.2.2 for $\mathcal{A}_\epsilon$.) We apply the reverse log-Sobolev inequality in [7], i.e.,

$$tP_t f\Gamma(\ln P_t f) + \rho_2 t^2 P_t f \Gamma^Z(\ln P_t f) \leq \left(1 + \frac{2\kappa}{\rho_2} + 2Kt\right)(P_t(f \ln f) - (P_t f) \ln P_t f).$$

Then our desired inequality is instantly deduced by $P_t(f \ln f) \leq (P_t f) \ln M$ and $\rho_2 t^2 P_t f \Gamma^Z \geq 0$. ■

Lemma 3.5.2 *If $\mathbb{M}$ satisfies $CD(-K, \rho_2, \kappa, d)$, there exists $C_h = C_h(\rho_2, \kappa, d) > 0$, $t > 0, x, y \in \mathbb{M}$,*

$$\Gamma_x(\ln p_t(x,y)) \leq \frac{C_h}{t}\left(1 + \frac{2\kappa}{\rho_2} + Kt\right)\left(K(t + d(x,y)^2) + \frac{d(x,y)^2}{t}\right).$$

Proof Let $t > 0$ and $y \in \mathbb{M}$. Let $u(x, s) := p_{\frac{t}{2}+s}(x, y)$, then u is a smooth, positive solution to the heat equation. From the heat kernel upper bound (3.12), for $t_0 = \frac{1}{6C_6K}$, $0 < t < t_0$, $0 \leq s \leq \frac{t}{2}$, $\forall x \in \mathbb{M}$,

$$\begin{aligned} u(x,s) &\leq \frac{C_5}{\mu(B(y, \sqrt{\frac{t}{2} + s}))} \exp\left(C_6 K(\frac{t}{2} + s + d(x,y)^2) - \frac{d(x,y)^2}{6(\frac{t}{2} + s)}\right) \\ &\leq \frac{C_6 e^{1/6}}{\mu(B(y, \sqrt{\frac{t}{2} + s}))} \leq \frac{C_6'}{\mu(B(y, \sqrt{\frac{t}{2}}))} = M. \end{aligned}$$

Moreover $u(x, s) \leq M$ for all $s > 0$, since $\|P_t\|_{\infty \to \infty} \leq 1$ for any $t > 0$.

By the Hamilton's inequality (3.18), the heat kernel lower bound (3.8) for $u(x, s)$ with $s = \frac{t}{2}$ and the doubling property (3.11),

$$\begin{aligned} \frac{t}{2}\Gamma(\ln u(x, \frac{t}{2})) &\leq \left(1 + \frac{2\kappa}{\rho_2} + Kt\right) \ln\left(\frac{M}{u(x, \frac{t}{2})}\right) \\ &\leq \left(1 + \frac{2\kappa}{\rho_2} + Kt\right) \frac{C_h}{2}\left(K(t + d(x,y)^2) + \frac{d(x,y)^2}{t}\right), \end{aligned}$$

where $C_h = 2\ln\left(C_6' C_1^{-1} C_{d1} 2^{Q/2}\right) \max(\frac{D}{2d}, \frac{4C_{d2}}{3} + C_2)$. ■

If we combine the previous lemma with (3.12), we obtain the following simpler statement for small t, which will be useful in the next section.

Lemma 3.5.3 *Assume $CD(-K, \rho_2, \kappa, d)$. For any $R > 0$, $\beta > 0$ and $x_0 \in \mathbb{M}$, there exists $C > 0$, $t_0 > 0$ such that for $d(x, y) \geq R/4$, $0 < t < t_0$,*

$$\sqrt{\Gamma_y(p_t(x, y))} \leq \frac{Ce^{-\beta R^2}}{\sqrt{t}\mu(B(x, \sqrt{t}))}.$$

3.5.3 Proof of Theorem 3.1.3, $p = 1$

Prior to the uniqueness of L^1 solution for the heat equation, we prove the uniqueness of L^1 harmonic function. Basic idea of the proof comes from [51].

Remark 3.5.4 *We assume the fixed curvature bound $\rho_1 = -K$ instead of the negative quadratic lower bound of Ricci curvature of [51].*

Theorem 3.5.5 *If $\mathbb{M}$ satisfies $CD(-K, \rho_2, \kappa, d)$, then any L^1 non-negative subharmonic function on $\mathbb{M}$ must be identically constant.*
In particular, any L^1 harmonic function on $\mathbb{M}$ must be identically constant.

Proof Let $g \in L^1(\mathbb{M})$ be a non-negative function satisfying $Lg \geq 0$, i.e. subharmonic. For any $t > 0$,

$$\begin{aligned} LP_t g(x) &= \int_{\mathbb{M}} (L_x p_t(x, y)) g(y) d\mu(y) \\ &= \int_{\mathbb{M}} (\frac{\partial}{\partial t} p_t(x, y)) g(y) d\mu(y) = \int_{\mathbb{M}} (L_y p_t(x, y)) g(y) d\mu(y). \end{aligned}$$

We claim the following integration by parts.

$$\int_{\mathbb{M}} (L_y p_t(x, y)) g(y) d\mu(y) = \int_{\mathbb{M}} p_t(x, y) Lg(y) d\mu(y). \tag{3.19}$$

To justify the claim, we observe the following

$$
\begin{aligned}
&\left|\int_{\mathbb{M}} \psi_R(y)\big[g(y)L_y p_t(x,y) - p_t(x,y)Lg(y)\big]d\mu(y)\right| \\
&= \left|\int_{\mathbb{M}} -\big[\Gamma(\psi_R g, p_t(x,\cdot)) - \Gamma(\psi_R p_t(x,\cdot), g)\big]d\mu\right| \\
&= \left|\int_{\mathbb{M}} -\big[g\Gamma(\psi_R, p_t(x,\cdot)) - p_t(x,\cdot)\Gamma(\psi_R, g)\big]d\mu\right| \\
&\le \int_{B(x_0,R+1)\setminus B(x_0,R)} C\left(g\sqrt{\Gamma(p_t(x,\cdot))} + p_t(x,\cdot)\sqrt{\Gamma(g)}\right)d\mu,
\end{aligned}
\tag{3.20}
$$

where $\psi_R \ge 0$ is a Lipschitz continuous cut-off function satisfying $\psi_R|_{B(x_0,R)} = 1$, $\operatorname{supp}\psi_R \subset B(x_0, R+1)$ and $\sqrt{\Gamma(\psi_R)} \le C$ almost everywhere for some $C > 0$ which is independent to $R > 0$.(See Remark 3.3.4.)

It suffices to show that both integrals on the right-hand side vanish as $R \to \infty$. We can consider R large enough so that $x \in B(x_0, R/4)$.

Let φ be a cut-off function for an annulus satisfying $\varphi|_{B(x_0,R+1)\setminus B(x_0,R)} = 1$, $\varphi|_{B(x_0,R-1)\cup(\mathbb{M}\setminus B(x_0,R+2))} = 0$ and $\sqrt{\Gamma(\varphi)} \le C$ almost everywhere. By the subharmonicity of g,

$$
\begin{aligned}
0 &\le \int_{\mathbb{M}} \varphi^2 gLg d\mu = -2\int_{\mathbb{M}} \varphi g\Gamma(\varphi, g)d\mu - \int_{\mathbb{M}} \varphi^2\Gamma(g)d\mu \\
&\le \int_{\mathbb{M}} \left[-\frac{1}{2}(2g\sqrt{\Gamma(\varphi)} - \varphi\sqrt{\Gamma(g)})^2 + 2\Gamma(\varphi)g^2 - \frac{1}{2}\varphi^2\Gamma(g)\right]d\mu \\
&\le 2\int_{\mathbb{M}} \Gamma(\varphi)g^2 d\mu - \frac{1}{2}\int_{\mathbb{M}} \varphi^2\Gamma(g)d\mu.
\end{aligned}
$$

Therefore, applying L^1 mean value estimate Corollary 3.3.6 to g,

$$
\begin{aligned}
&\int_{B(x_0,R+1)\setminus B(x_0,R)} \Gamma(g)d\mu \le 4\int_{\mathbb{M}} \Gamma(\varphi)g^2 d\mu \le 4C^2\int_{B(x_0,R+2)} g^2 d\mu \\
&\le 4C^2\|g\|_{L^1} \sup_{B(x_0,R+2)} g(y) \le C'e^{c'KR^2}\frac{1}{\mu(B(x_0,2R+4))}\|g\|_{L^1}^2.
\end{aligned}
$$

By Schwarz inequality,

$$
\begin{aligned}
\int_{B(x_0,R+1)\setminus B(x_0,R)} \sqrt{\Gamma(g)}d\mu &\le \left(\int_{B(x_0,R+1)\setminus B(x_0,R)} \Gamma(g)d\mu\right)^{\frac{1}{2}} (\mu(B(x_0,2R+4)))^{\frac{1}{2}} \\
&= Ce^{cKR^2}\|g\|_{L^1}.
\end{aligned}
$$

In addition to this estimate, to bound the second integration in (3.20) we consider the upper bound of heat kernel (3.12):

$$p_t(x,y) \le \frac{C_5}{\mu(B(x,\sqrt{t}))} \exp\left(C_6 K(t + d(x,y)^2) - \frac{d(x,y)^2}{6t}\right).$$

Combining the above two inequalities, we estimate the second term of (3.20) for small $0 < t < T = T(K, \rho_2, \kappa, d)$.

$$\begin{aligned}
&\int_{B(x_0,R+1)\setminus B(x_0,R)} p_t(x,\cdot)\sqrt{\Gamma(g)}d\mu \\
&\quad \le \left(\sup_{y\in B(x_0,R+1)\setminus B(x_0,R)} p_t(x,y)\right)\int_{B(x_0,R+1)\setminus B(x_0,R)} \sqrt{\Gamma(g)}d\mu \\
&\quad \le \frac{C}{\mu(B(x,\sqrt{t}))}\exp\left(-\alpha\frac{R^2}{t}\right)\|g\|_{L^1} \xrightarrow{R\to\infty} 0,
\end{aligned}$$

with $d(x_0,x) \le \frac{R}{4}$, $\frac{R}{2} \le d(x,y) \le 4R$ and $\alpha > 0$.

For the first term of (3.20), L^1 mean value estimate for g yields

$$\begin{aligned}
&\int_{B(x_0,R+1)\setminus B(x_0,R)} g\sqrt{\Gamma(p_t(x,\cdot))}d\mu \\
&\quad \le \left(\sup_{B(x_0,R+1)\setminus B(x_0,R)} g\right)\int_{B(x_0,R+1)\setminus B(x_0,R)} \sqrt{\Gamma(p_t(x,\cdot))}d\mu \\
&\quad \le \left(\frac{Ce^{cKR^2}}{\mu(B(x_0,2R+2))}\|g\|_{L^1}\right)\int_{B(x_0,R+1)\setminus B(x_0,R)} \sqrt{\Gamma(p_t(x,\cdot))}d\mu.
\end{aligned}$$

If we apply Lemma 3.5.3 for $\beta > cK$,

$$\le \left(\frac{\mu(B(x_0,R+1))}{\mu(B(x_0,2R+2))}\|g\|_{L^1}\right)\frac{Ce^{-\beta' R^2}}{\sqrt{t}\mu(B(x,\sqrt{t}))} \le \|g\|_{L^1}\frac{Ce^{-\beta' R^2}}{\sqrt{t}\mu(B(x,\sqrt{t}))} \xrightarrow{R\to\infty} 0,$$

where $0 < t < T = T(K, \rho_2, \kappa, d)$ small enough and $\beta' > 0$.

Therefore, as $R \to \infty$, the integration of (3.20) vanishes as we desired, and we proved our claim (3.19).

Now since the integration by part (3.19) holds for small t, we have

$$\frac{\partial}{\partial t}P_t g = LP_t g = P_t(Lg) \ge 0.$$

And by the semigroup property, $P_t g(x) \geq g(x)$ for all $t > 0$, $x \in \mathbb{M}$.

On the other hand, by the stochastic completeness ([10]) of P_t, $\|P_t g\|_{L^1} = \|g\|_{L^1}$. Therefore $P_t g = g$, i.e. g is harmonic.

For any constant $\gamma > 0$, $(g-\gamma)_+ = \max(0, g-\gamma) \leq g$ is also a non-negative L^1 subharmonic function. And by the same argument, it is harmonic. $\min(g,\gamma) = g - (g-\gamma)_+$ is also non-negative L^1 harmonic function. Observe that $\min(g,\gamma) \in C^\infty(\mathbb{M})$ for any $\gamma > 0$ by the hypoellipticity of L. This is not possible unless g is constant.

Finally, any harmonic function $u \in L^1(\mathbb{M})$ is identically constant since $|u|$ is non-negative L^1 subharmonic function which must be constant by the above. ■

With the uniqueness of L^1 harmonic function, we are ready to prove L^1 uniqueness of the solution for the subelliptic heat equation.

Theorem 3.5.6 *Let* $\mathbb{M}$ *satisfy* $CD(-K, \rho_2, \kappa, d)$. *Let* $v : \mathbb{M} \times [0,\infty) \to \mathbb{R}$ *be a non-negative function satisfying*

$$\left(L - \frac{\partial}{\partial t}\right) v(x,t) \geq 0, \qquad \|v(\cdot,t)\|_{L^1(\mathbb{M})} < \infty, \quad \forall t > 0,$$
$$\|v(\cdot,t)\|_{L^1(\mathbb{M})} \xrightarrow{t\to 0} 0,$$

then $v(x,t) \equiv 0$ *on* $\mathbb{M} \times (0,\infty)$.

Proof For any $\epsilon > 0$, denote

$$\psi_\epsilon(x,t) = \max\big(0, v(x,t+\epsilon) - P_t(v(\cdot,\epsilon))\big).$$

Then it follows that $\psi_\epsilon \geq 0$, $\lim_{t\to 0} \psi_\epsilon(x,t) = 0$, $\left(L - \frac{\partial}{\partial t}\right) \psi_\epsilon \geq 0$.

Fix $T > 0$. Define

$$f(x) = \int_0^T \psi_\epsilon(x,t)dt,$$

which satisfies $Lf(x) = \psi_\epsilon(x,T) - \psi_\epsilon(x,0) \geq 0$.

The assumption $v(\cdot,t) \in L^1(\mathbb{M})$ yields $\int_0^T \int_{\mathbb{M}} |v(x,t+\epsilon)| d\mu(x)dt < \infty$. Together with $\int_0^T \int_{\mathbb{M}} P_t v(x,\epsilon) d\mu dt \leq T \int_{\mathbb{M}} v(x,\epsilon) d\mu(x) < \infty$, we obtain $\|f\|_{L^1(\mathbb{M})} < \infty$.

Now f is non-negative L^1 subharmonic function, so that we can apply Theorem 3.5.5 to f and conclude f is identically constant. This implies $0 = Lf(\cdot) = \psi_\epsilon(\cdot, T)$ for arbitrary $T > 0$. Hence for any $t > 0$,

$$\begin{aligned} v(x, t+\epsilon) &\leq P_t(v(\cdot, \epsilon))(x) \\ &\leq \|p_t(x, \cdot)\|_\infty \|v(\cdot, \epsilon)\|_{L^1} \leq M\|v(\cdot, \epsilon)\|_{L^1} \xrightarrow{\epsilon \to 0} 0, \end{aligned}$$

where the uniform bound for $\|p_t(x, \cdot)\|_\infty$ is found in Lemma 3.5.2.

Therefore non-negative $v(x, t)$ must be zero for all $(x, t) \in \mathbb{M} \times (0, \infty)$. ■

Proof [Proof of Theorem 3.1.3, $p = 1$]

For any L^1 solution u of $\left(L - \frac{\partial}{\partial t}\right) u = 0$ with the initial condition $u \xrightarrow{L^1} f \in L^1(\mathbb{M})$ as $t \to 0$,

$$v(x, t) := |u(x, t) - P_t f(x)|$$

will be a non-negative L^1 subsolution of the heat equation with $v \xrightarrow{L^1} 0$ as $t \to 0$.

By the previous theorem, $v \equiv 0$ on $\mathbb{M} \times (0, \infty)$. Therefore, u is uniquely determined to be $P_t f$.

■

Remark 3.5.7 *In [2], [1], the measure contractive definition of Ricci tensor bound and volume comparison theorem (which is not yet established in our framework) are introduced in three dimensional sub-Riemannian spaces. This measure contraction property is extended to higher dimensions in [46].*

4. LOWER BOUNDS OF THE FIRST EIGENVALUE OF SUB-LAPLACIAN

4.1 Introduction

The study of optimal lower bounds for sub-Laplacians on manifolds has attracted a lot of interest in the past few years. In particular, the most studied example has been the example of the sub-Laplacian on CR manifolds. In that case, the story goes back at least to the work by Greenleaf [31] which has seen, since then, several improvements and variations. Some optimal lower bounds for the first eigenvalue of sub-Laplacians also have been obtained in the context of quaternionic contact manifolds by Ivanov-Petkov-Vassilev [37]. More general situations were even considered by Hladky [34].

In the present work, we obtain optimal first eigenvalue lower bounds in a large class of sub-Riemannian manifolds that encompasses as a very special case Sasakian manifolds and 3-Sasakian manifolds. This class is the class of sub-Riemannian manifolds with transverse symmetries that was introduced in [10]. Roughly speaking, a sub-Riemannian manifold with transverse symmetries is a sub-Riemannian manifold for which the horizontal distribution admits a canonical intrinsic complement which is generated by sub-Riemannian Killing fields. The lower bound we obtain in that case improves a previous lower bound that was obtained by Baudoin-Kim in [12]. The method of [12] was to apply to an eigenfunction of the sub-Laplacian the curvature-dimension inequality proved in [10], and then to integrate this curvature-dimension inequality over the manifold. When used on a Riemannian manifold, this technique provides the optimal Lichnerowicz estimate. However, interestingly, this technique does not give the optimal estimate in the sub-Riemannian case and more work is needed. Our approach here, is to take advantage of the Bochner-Weitzenböck formula that was recently proved in [6] and to integrate this equality over the manifold. This gives an equality which when applied to an eigenfunction gives a better estimate than [12] for the first eigenvalue. In the 1 or the 3-Sasakian case, the lower bound we obtain coincides with the known optimal lower bound.

In the second part of the article, we check the optimality of our lower bound, by proving a rigidity result in the spirit of Obata [55]. More precisely we prove the following result:

Theorem 4.1.1 *Let $\mathbb{M}$ be a compact sub-Riemannian manifold of H-type with dimension $d+\mathfrak{h}$, d being the dimension of the horizontal bundle and $\mathfrak{h}$ the dimension of the vertical bundle. Assume that for every smooth horizontal one-form η,*

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho \|\eta\|^2_{\mathcal{H}^*},$$

with $\rho > 0$, then the first eigenvalue λ_1 of the sub-Laplacian $-L$ satisfies

$$\lambda_1 \geq \frac{\rho d}{d-1+3\mathfrak{h}}.$$

Moreover, if $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$, then $\mathbb{M}$ is equivalent to a 1-Sasakian sphere $\mathbb{S}^{2m+1}(r)$ or a 3-Sasakian sphere $\mathbb{S}^{4m+3}(r)$ for some $r > 0$ and $m \geq 1$.

This result for H-type manifolds generalizes the corresponding theorem for Sasakian manifolds by Chang-Chiu [22] and for 3-Sasakian manifolds by Ivanov-Petkov-Vassilev [37]. Like in the cited references, the main idea is to prove that an extremal eigenfunction f for the sub-Laplacian needs to satisfy $\tilde{\nabla}^2 f = -\alpha f$, for the Levi-Civita connection of a well chosen Riemannian extension of the sub-Riemannian metric. We can observe that in the works [38, 51] the Sasakian condition is not needed, it is therefore an interesting question to try to generalize our result to more general sub-Riemannian structures where the transverse symmetries condition is not assumed.

This chapter is organized as follows. Section 2 presents the basic materials on sub-Riemannian manifolds with transverse symmetries. In particular, we present the Bochner-Weitzenböck formula that was proved in [6]. Section 3 is devoted to the proof of the lower bound for the first eigenvalue and Section 4 proves its optimality in the context of H-type manifolds.

4.2 The Bochner-Weitzenböck formula on sub-Riemannian manifolds with transverse symmetries

The notion of sub-Riemannian manifold with transverse symmetries was introduced in [10]. We recall here the main geometric quantities and operators related to this structure and we refer to [6] and [10] for further details. We in particular focus on the Bochner-Weitzenböck formula that was proved in [6].

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $d+\mathfrak{h}$. We assume that $\mathbb{M}$ is equipped with a bracket generating distribution $\mathcal{H}$ of dimension d and a fiberwise inner product $g_{\mathcal{H}}$ on that distribution. The distribution $\mathcal{H}$ is referred to as the set of *horizontal directions*, while a vector field which is tangent to $\mathcal{H}$ is said to be horizontal.

Definition 4.2.1 *It is said that $\mathbb{M}$ is a sub-Riemannian manifold with transverse symmetries if there exists an $\mathfrak{h}$- dimensional Lie algebra $\mathcal{V}$ of sub-Riemannian Killing vector fields such that for every $x \in \mathbb{M}$,*

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x).$$

We recall that a vector field Z is said to be a sub-Riemannian Killing vector field if the flow it generates locally preserves the horizontal distribution and induces a $g_{\mathcal{H}}$-isometry. Also $\mathcal{V}$ denotes the distribution referred to as the set of *vertical directions.* The choice of an inner product $g_{\mathcal{V}}$ on the Lie algebra $\mathcal{V}$ naturally endows $\mathbb{M}$ with a one-parameter family of Riemannian metrics that makes the decomposition $\mathcal{H} \oplus \mathcal{V}$ orthogonal:

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0.$$

For notational convenience, we will often use the notation $\langle \cdot, \cdot \rangle_\varepsilon$, resp. $\langle \cdot, \cdot \rangle_\mathcal{H}$, resp $\langle \cdot, \cdot \rangle_\mathcal{V}$, instead of g_ε, resp. $g_\mathcal{H}$, resp. $g_\mathcal{V}$. We can extend $g_\mathcal{H}$ on $T_x\mathbb{M} \times T_x\mathbb{M}$ by the requirement that $g_\mathcal{H}(u,v) = 0$ whenever u or v is in $\mathcal{V}(x)$. We similarly extend $g_\mathcal{V}$. Hence for any $u \in T_x\mathbb{M}$,

$$\|u\|_\varepsilon^2 = \|u\|_\mathcal{H}^2 + \frac{1}{\varepsilon}\|u\|_\mathcal{V}^2.$$

The volume measure obtained as a product of the horizontal volume measure determined by $g_\mathcal{H}$ and the volume measure determined by $g_\mathcal{V}$ will be denoted by μ and is our reference measure on $\mathbb{M}$.

The following connection was introduced in [10].

Proposition 4.2.1 (See [10]) *There exists a unique connection ∇ on $\mathbb{M}$ satisfying the following properties:*

(i) $\nabla g_\varepsilon = 0$, $\varepsilon > 0$;

(ii) If X and Y are horizontal vector fields, $\nabla_X Y$ is horizontal;

(iii) If $Z \in \mathcal{V}$, $\nabla Z = 0$;

(iv) If X, Y are horizontal vector fields and $Z \in \mathcal{V}$, the torsion vector field $T(X,Y)$ is vertical and $T(X,Z) = 0$.

Intuitively ∇ is the connection which coincides with the Levi-Civita connection of the Riemannian metric g_ε on the horizontal bundle $\mathcal{H}$ and that parallelizes the Lie algebra $\mathcal{V}$.

At every point $x \in \mathbb{M}$, we can find a local frame of vector fields $\{X_1, \cdots, X_d, Z_1, \cdots, Z_\mathfrak{h}\}$ such that on a neighborhood of x:

(a) $\{X_1, \cdots, X_d\}$ is a $g_\mathcal{H}$-orthonormal basis of $\mathcal{H}$;

(b) $\{Z_1, \cdots, Z_\mathfrak{h}\}$ is a $g_\mathcal{V}$-orthonormal basis of the Lie algebra $\mathcal{V}$;

Such a frame will be called a local adapted frame.

The sub-Laplacian on $\mathbb{M}$ is the second-order differential operator which is given in a local adapted frame by

$$L = \sum_{i=1}^{d} \nabla_{X_i}\nabla_{X_i} - \nabla_{\nabla_{X_i}X_i}. \tag{4.1}$$

By declaring a one-form horizontal (resp. vertical) if it vanishes on the vertical bundle $\mathcal{V}$ (resp. on the horizontal bundle $\mathcal{H}$), the splitting of the tangent space

$$T_x\mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x)$$

gives a splitting of the cotangent space

$$T_x^*\mathbb{M} = \mathcal{H}^*(x) \oplus \mathcal{V}^*(x).$$

If $\{X_1, \cdots, X_d, Z_1, \cdots, Z_{\mathfrak{h}}\}$ is a local adapted frame, the dual frame will be denoted $\{\theta_1, \cdots, \theta_d, \nu_1, \cdots, \nu_{\mathfrak{h}}\}$ and referred to as a local adapted coframe. With a slight abuse of notations, for $\varepsilon > 0$, the metric on $T_x^*\mathbb{M}$ that makes $\{\theta_1, \cdots, \theta_d, \frac{1}{\sqrt{\varepsilon}}\nu_1, \cdots, \frac{1}{\sqrt{\varepsilon}}\nu_{\mathfrak{h}}\}$ orthonormal will still be denoted g_ε or $\langle \cdot, \cdot \rangle_\varepsilon$. This metric on the cotangent bundle can thus be written

$$g_\varepsilon = g_{\mathcal{H}^*} \oplus \varepsilon g_{\mathcal{V}^*}, \quad \varepsilon > 0, \tag{4.2}$$

where $g_{\mathcal{H}^*}$ (resp. $g_{\mathcal{V}^*}$) is the metric on $\mathcal{H}^*$ (resp. $\mathcal{V}^*$) that makes $\{\theta_1, \cdots, \theta_d\}$ (resp. $\{\nu_1, \cdots, \nu_{\mathfrak{h}}\}$) orthonormal. We use similar notations and conventions as before so that for every η in $T_x^*\mathbb{M}$,

$$\|\eta\|_\varepsilon^2 = \|\eta\|_{\mathcal{H}^*}^2 + \varepsilon \|\eta\|_{\mathcal{V}^*}^2.$$

We now introduce some tensors that will play an important role in the sequel. We define $\mathfrak{Ric}_{\mathcal{H}} : T_x^*\mathbb{M} \to T_x^*\mathbb{M}$ as the symmetric linear map on one forms such that for every smooth functions f, g,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg \rangle_{\mathcal{H}^*} = \mathbf{Ricci}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g),$$

where **Ricci** is the Ricci curvature of the connection ∇ and $\nabla_{\mathcal{H}}$ the horizontal gradient (projection of the gradient on the horizontal distribution $\mathcal{H}$). Similarly, we will denote by $\nabla_{\mathcal{V}}$ the vertical gradient, that is the projection of the gradient on the vertical bundle. In a local adapted frame $\{X_1, \cdots, X_d, Z_1, \cdots, Z_{\mathfrak{h}}\}$, we have thus

$$\nabla_{\mathcal{H}} f = \sum_{i=1}^{d} (X_i f) X_i,$$

$$\nabla_{\mathcal{V}} f = \sum_{m=1}^{\mathfrak{h}} (Z_m f) Z_m.$$

and

$$\mathbf{Ricci}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g) = \sum_{n=1}^{d} g_{\mathcal{H}}(\mathbf{R}(\nabla_{\mathcal{H}} f, X_i) X_i, \nabla_{\mathcal{H}} g),$$

where $\mathbf{R}$ is the Riemanna curvature tensor: $\mathbf{R}(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i,X_j]}X_k$.

For $Z \in \mathcal{V}$, we consider the unique skew-symmetric map J_Z defined on the horizontal bundle $\mathcal{H}$ such that for every horizontal vector fields X and Y,

$$\langle J_Z(X), Y \rangle_{\mathcal{H}} = \langle Z, T(X,Y) \rangle_{\mathcal{V}}. \tag{4.3}$$

We can then extend J_Z to the whole tangent space $T_x\mathbb{M}$ by imposing that $J_Z(V) = 0$ whenever V is a vertical vector field. If $(Z_m)_{1 \le m \le \mathfrak{h}}$ is a $g_{\mathcal{V}}$-orthonormal basis of the Lie algebra $\mathcal{V}$, the operator $\sum_{m=1}^{\mathfrak{h}} J_{Z_m}^* J_{Z_m} = -\sum_{m=1}^{\mathfrak{h}} J_{Z_m}^2 : T_x\mathbb{M} \to T_x\mathbb{M}$ does not depend on the choice of the basis and will concisely be denoted by $-\mathbf{J}^2$. We can note that in the case where $\mathbb{M}$ is a Sasakian manifold, like the Heisenberg group for instance, $-\mathbf{J}^2$ is the identity map on the horizontal distribution. Though originally defined on vector fields we will also consider $-\mathbf{J}^2$ as the linear map $T_x^*\mathbb{M} \to T_x^*\mathbb{M}$ defined by

$$\langle -\mathbf{J}^2(\theta_i), \theta_j \rangle_{\mathcal{H}^*} = \langle -\mathbf{J}^2(X_i), X_j \rangle_{\mathcal{H}}, \qquad 1 \le i, j \le d$$

Then $-\mathbf{J}^2$ is defined to be 0 on vertical one-forms.

If V is a horizontal vector field, then we consider an operator $\mathfrak{T}_V^\varepsilon$ on smooth sections of the cotangent bundle given by

$$\mathfrak{T}_V^\varepsilon \eta = -\sum_{j=1}^{d} \eta(T(V,X_j))\theta_j + \frac{1}{2\varepsilon}\sum_{m=1}^{\mathfrak{h}} \eta(J_{Z_m}V)\nu_m$$

in a local frame. It is easily seen that $\mathfrak{T}_V^\varepsilon$ is a skew-symmetric operator for the metric $g_{2\varepsilon}$ that was previously defined on one-forms by (4.2).

If η is a one-form, we define the horizontal gradient in a local adapted frame of η as the $(0,2)$ tensor

$$\nabla_{\mathcal{H}}\eta = \sum_{i=1}^{d} \nabla_{X_i}\eta \otimes \theta_i.$$

Similarly, we will use the notation

$$\mathfrak{T}_{\mathcal{H}}^\varepsilon \eta = \sum_{i=1}^{d} \mathfrak{T}_V^\varepsilon \eta \otimes \theta_i.$$

We finally recall the following definition that was introduced in [10]:

Definition 4.2.2 *The sub-Riemannian manifold $\mathbb{M}$ is said to be of Yang-Mills type, if the horizontal divergence of the torsion vanishes that is for every horizontal vector field X, and every adapted local frame*

$$\sum_{\ell=1}^{d} (\nabla_{X_\ell} T)(X_\ell, X) = 0.$$

There are many interesting examples of Yang-Mills sub-Riemannian manifolds with transverse symmetries (see [10]). Sasakian and 3-Sasakian manifolds are examples of Yang-Mills sub-Riemannian manifolds. Though not identical, the Yang-Mills condition can be compared to the divergence free torsion condition that was considered in [38].

The following Bochner Weitzenböck formula was proved in [6] to which we refer for further details.

Theorem 4.2.1 (Bochner-Weitzenböck formula [6]) *Assume that* $\mathbb{M}$ *is a sub-Riemannian manifold with transverse symmetries of Yang-Mills type. For* $\varepsilon > 0$*, we consider the* $g_{2\varepsilon}$*-self-adjoint operator which is defined on one-forms by*

$$\square_\varepsilon = -(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H})^*(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H}) - \frac{1}{2\varepsilon}\mathbf{J}^2 - \mathfrak{Ric}_\mathcal{H}.$$

Then, for every smooth function f *on* $\mathbb{M}$,

$$d(Lf) = \square_\varepsilon(df),$$

and for any smooth one-form η,

$$\frac{1}{2}L\|\eta\|^2_{2\varepsilon} - \langle \square_\varepsilon \eta, \eta\rangle_{2\varepsilon} = \|\nabla_\mathcal{H}\eta - \mathfrak{T}^\varepsilon_\mathcal{H}\eta\|^2_{2\varepsilon} + \left\langle \mathfrak{Ric}_\mathcal{H}(\eta) + \frac{1}{2\varepsilon}\mathbf{J}^2(\eta), \eta \right\rangle_{\mathcal{H}^*}.$$

In the previous statement $(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H})^*$ is understood as an adjoint for the $g_{2\varepsilon}$-metric and it is easily seen (see [6]) that in a local adapted frame, we have

$$-(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H})^*(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H}) = \sum_{i=1}^d (\nabla_{X_i} - \mathfrak{T}^\varepsilon_{X_i})^2 - (\nabla_{\nabla_{X_i}X_i} - \mathfrak{T}^\varepsilon_{\nabla_{X_i}X_i}),$$

and for any smooth one-form η,

$$\|\nabla_\mathcal{H}\eta - \mathfrak{T}^\varepsilon_\mathcal{H}\eta\|^2_{2\varepsilon} = \sum_{i=1}^d \|\nabla_{X_i}\eta - \mathfrak{T}^\varepsilon_{X_i}\eta\|^2_{2\varepsilon}.$$

4.3 Lichnerowicz estimate

From now on, we consider a compact Yang-Mills sub-Riemannian manifold $\mathbb{M}$ with transverse symmetries and adopt the conventions and notations of the previous section. In particular L denotes the sub-Laplacian on $\mathbb{M}$. In this section, we prove the following result.

Theorem 4.3.1 *Assume that for every smooth horizontal one-form* η,

$$\langle \mathfrak{Ric}_\mathcal{H}(\eta), \eta\rangle_{\mathcal{H}^*} \ge \rho_1\|\eta\|^2_{\mathcal{H}^*}, \quad \left\langle -\mathbf{J}^2(\eta), \eta\right\rangle_{\mathcal{H}^*} \le \kappa\|\eta\|^2_{\mathcal{H}^*},$$

and that for every $Z \in \mathcal{V}$,

$$\mathbf{Tr}(J_Z^* J_Z) \ge \rho_2\|Z\|^2_\mathcal{V},$$

with $\rho_1, \rho_2 > 0$ and $\kappa \geq 0$. Then the first eigenvalue λ_1 of the sub-Laplacian $-L$ satisfies

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{d} + \frac{3\kappa}{\rho_2}}.$$

Before we prove the result, we briefly discuss the argument that was used in [12] to quickly get, under the same assumptions, a lower bound on λ_1 which is less sharp.

If f is a smooth function on $\mathbb{M}$, then we have from Theorem 4.2.1

$$\frac{1}{2}L\|df\|_{2\varepsilon}^2 - \langle d(Lf), df\rangle_{2\varepsilon} = \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left\langle \mathfrak{Ric}_{\mathcal{H}}(df) + \frac{1}{2\varepsilon}\mathbf{J}^2(df), df \right\rangle_{\mathcal{H}^*}.$$

Integrating this equality over $\mathbb{M}$ and using the assumptions

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta\rangle_{\mathcal{H}^*} \geq \rho_1 \|\eta\|_{\mathcal{H}^*}^2, \quad \left\langle -\mathbf{J}^2(\eta), \eta\right\rangle_{\mathcal{H}^*} \leq \kappa \|\eta\|_{\mathcal{H}^*}^2,$$

we deduce

$$-\int_{\mathbb{M}} \langle d(Lf), df\rangle_{2\varepsilon} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2.$$

An integration by parts of left hand side of the inequality gives then

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df\rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{2\varepsilon}^2 + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2. \tag{4.4}$$

Now, a straightforward application of the Cauchy-Schwarz inequality yields the pointwise lower bound

$$\|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\mathcal{H}^*}^2 \geq \frac{1}{d}(Lf)^2 + \frac{1}{4}\rho_2 \|df\|_{\mathcal{V}^*}^2. \tag{4.5}$$

Coming back to (4.4), we infer then

$$\frac{d-1}{d} \int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df\rangle_{\mathcal{V}^*} \geq \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|_{\mathcal{H}^*}^2 + \frac{1}{4}\rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

In particular, if $Lf = -\lambda_1 f$, then we obtain

$$\frac{d-1}{d}\lambda_1^2 \int_{\mathbb{M}} f^2 + 2\varepsilon\lambda_1 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2 \geq \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \lambda_1 \int_{\mathbb{M}} f^2 + \frac{1}{4}\rho_2 \int_{\mathbb{M}} \|df\|_{\mathcal{V}^*}^2.$$

Choosing ε such that $2\varepsilon\lambda_1 = \frac{1}{4}\rho_2$ yields

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{d} + \frac{4\kappa}{\rho_2}}.$$

This is not the optimal lower bound we are looking for. It is possible to improve this lower bound from (4.4) by first integrating by parts the term $\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon}$ and, then using Cauchy-Schwarz inequality. The key lemma is the following:

Lemma 4.3.2 *For $f \in C^\infty(\mathbb{M})$,*

$$\begin{aligned}\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} = &\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} + 2\varepsilon \int_{\mathbb{M}} \left\|\nabla_{\mathcal{H}} df - \frac{3}{2}\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\right\|^2_{\mathcal{V}^*} \\ &+ \frac{1}{2}\int_{\mathbb{M}} \mathbf{Tr}(J^*_{\nabla_{\mathcal{V}} f} J_{\nabla_{\mathcal{V}} f}) - \frac{5}{2}\varepsilon \int_{\mathbb{M}} \|\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{V}^*}.\end{aligned}$$

Proof Using the definition $\mathfrak{T}^{\varepsilon}_{\mathcal{H}}$ together with the Yang-Mills assumption, we see that

$$\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} = \frac{1}{4\varepsilon}\int_{\mathbb{M}} \mathbf{Tr}(J^*_{\nabla_{\mathcal{V}} f} J_{\nabla_{\mathcal{V}} f}). \tag{4.6}$$

As a consequence, we obtain

$$\begin{aligned}&\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} \\ =&\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} + 2\varepsilon \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{V}^*} \\ =&\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} + 2\varepsilon \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df\|^2_{\mathcal{V}^*} - 4\varepsilon \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} + 2\varepsilon \int_{\mathbb{M}} \|\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{V}^*}\end{aligned} \tag{4.7}$$

By using (4.6), the trick is now to write

$$\begin{aligned}\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} &= \frac{3}{2}\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} - \frac{1}{2}\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} \\ &= \frac{3}{2}\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} df, \mathfrak{T}^{\varepsilon}_{\mathcal{H}}(df)\rangle_{\mathcal{V}^*} - \frac{1}{8\varepsilon}\int_{\mathbb{M}} \mathbf{Tr}(J^*_{\nabla_{\mathcal{V}} f} J_{\nabla_{\mathcal{V}} f}).\end{aligned}$$

Coming back to (4.7) and completing the squares gives

$$\begin{aligned}\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} = &\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} + 2\varepsilon \int_{\mathbb{M}} \left\|\nabla_{\mathcal{H}} df - \frac{3}{2}\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\right\|^2_{\mathcal{V}^*} \\ &+ \frac{1}{2}\int_{\mathbb{M}} \mathbf{Tr}(J^*_{\nabla_{\mathcal{V}} f} J_{\nabla_{\mathcal{V}} f}) - \frac{5}{2}\varepsilon \int_{\mathbb{M}} \|\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{V}^*}.\end{aligned}$$

■

We are now in position to complete the proof of Theorem 4.3.1.

Proof Using the previous Lemma, Cauchy-Schwarz inequality and the assumptions

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho_1 \|\eta\|^2_{\mathcal{H}^*}, \quad \left\langle -\mathbf{J}^2(\eta), \eta \right\rangle_{\mathcal{H}^*} \leq \kappa \|\eta\|^2_{\mathcal{H}^*}, \quad \mathbf{Tr}(J_Z^* J_Z) \geq \rho_2 \|Z\|^2_{\mathcal{V}}$$

we get the lower bound

$$\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} \geq \frac{1}{d} \int_{\mathbb{M}} (Lf)^2 + \frac{3}{4}\rho_2 \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*} - \frac{5}{8\varepsilon}\kappa \int_{\mathbb{M}} \|df\|^2_{\mathcal{H}^*}.$$

From (4.4), we know that

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|^2_{\mathcal{H}^*}. \tag{4.8}$$

We thus deduce

$$\frac{d-1}{d} \int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \left(\rho_1 - \frac{9\kappa}{8\varepsilon}\right) \int_{\mathbb{M}} \|df\|^2_{\mathcal{H}^*} + \frac{3}{4}\rho_2 \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*}.$$

Now if f satisfies $Lf = -\lambda_1 f$, we get

$$\frac{d-1}{d}\lambda_1^2 \int_{\mathbb{M}} f^2 + 2\varepsilon\lambda_1 \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*} \geq \left(\rho_1 - \frac{9\kappa}{8\varepsilon}\right) \lambda_1 \int_{\mathbb{M}} f^2 + \frac{3}{4}\rho_2 \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*}.$$

Choosing ε such that

$$2\varepsilon\lambda_1 = \frac{3}{4}\rho_2,$$

the desired lower bound on λ_1 is obtained. ■

4.4 The Obata sphere theorem on H-type manifolds

In this section we prove the optimality of the lower bound for the first eigenvalue of the sub-Laplacian on a special class of Yang-Mills manifolds by obtaining a rigidity result in the spirit of the Obata sphere theorem.

We first introduce the following definition inspired from the notion of H-type groups that was introduced by Kaplan.

Definition 4.4.1 *Let $\mathbb{M}$ be a sub-Riemannian manifold with transverse symmetries of Yang-Mills type. We will say that $\mathbb{M}$ is of H-type if for every $Z \in \mathcal{V}$, $\|Z\|_{\mathcal{V}} = 1$, the map J_Z is orthogonal, that is, $\langle J_Z(X), J_Z(Y)\rangle_{\mathcal{H}} = \langle X, Y\rangle_{\mathcal{H}}$ for $X, Y \in \mathcal{H}(x)$.*

Sasakian or 3-Sasakian manifolds are examples of H-type manifolds. If $\mathbb{M}$ is a H-type sub-Riemannian manifold, it is immediate from the definition that for $Z, Z' \in \mathcal{V}$,

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z, Z'\rangle_{\mathcal{V}} \mathbf{Id}_{\mathcal{H}}.$$

In particular, we have

$$J_Z^2 = -\|Z\|_{\mathcal{V}}^2 \mathbf{Id}_{\mathcal{H}}.$$

In this section, we prove the following result:

Theorem 4.4.1 *Let $\mathbb{M}$ be a compact sub-Riemannian manifold of H-type. Assume that for every smooth horizontal one-form η,*

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta\rangle_{\mathcal{H}^*} \geq \rho \|\eta\|_{\mathcal{H}^*}^2,$$

with $\rho > 0$, then the first eigenvalue λ_1 of the sub-Laplacian $-L$ satisfies

$$\lambda_1 \geq \frac{\rho d}{d - 1 + 3\mathfrak{h}}.$$

Moreover, if $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$, then $\mathbb{M}$ is equivalent to a 1-Sasakian sphere $\mathbb{S}^{2m+1}(r)$ or a 3-Sasakian sphere $\mathbb{S}^{4m+3}(r)$ for some $r > 0$ and $m \geq 1$.

To put things in perspective, we pause a little and describe the sub-Riemannian geometry of the 1 and 3 Sasakian spheres and precise what we mean by equivalent in the previous theorem.

- The sub-Riemannian geometry of the standard 1-Sasakian sphere $\mathbb{S}^{2m+1}(1)$ is induced from the Riemannian structure of the complex projective space $\mathbb{CP}^m$ by the Hopf fibration $\mathbf{U}(1) \to \mathbb{S}^{2m+1} \to \mathbb{CP}^m$. The sub-Laplacian L is then the lift of the Laplace-Beltrami operator on $\mathbb{CP}^m$. In that case, $\lambda_1 = 2m$.

- The sub-Riemannian geometry of the standard 3-Sasakian sphere $\mathbb{S}^{4m+3}$ is induced from the Riemannian structure of the quaternionic projective space $\mathbb{HP}^m$ by the quaternionic Hopf fibration $\mathbf{SU}(2) \to \mathbb{S}^{4m+3} \to \mathbb{HP}^m$. The sub-Laplacian L is then the lift of the Laplace-Beltrami operator on $\mathbb{HP}^n$. In that case, $\lambda_1 = m$.

In the previous theorem, we use the following notion of equivalence for sub-Riemannian manifolds with transverse symmetries: Two sub-Riemannian manifolds with transverse symmetries $(\mathbb{M}_1, \mathcal{H}_1, \mathcal{V}_1)$ and $(\mathbb{M}_2, \mathcal{H}_2, \mathcal{V}_2)$ are said to be equivalent if there exists a diffeomorphism $\mathbb{M}_1 \to \mathbb{M}_2$ that induces an isometry between the horizontal distributions $\mathcal{H}_1$ and $\mathcal{H}_2$ and a Lie algebra isomorphism between $\mathcal{V}_1$ and $\mathcal{V}_2$.

We now discuss the cases that were already known in the literature. As we pointed out Sasakian manifolds are of H-type. In that case $\mathfrak{h} = 1$ and the lower bound becomes

$$\lambda_1 \geq \frac{\rho d}{d+2}.$$

This estimate was obtained by Greenleaf [31] (see also [5]). The estimate is optimal and the corresponding Obata's type rigidity result was obtained in [22] (see also [38] and [51]).

The other case that was studied in the literature is the case of 3-Sasakian manifolds for which $\mathfrak{h} = 3$. The lower bound is then

$$\lambda_1 \geq \frac{\rho d}{d+8}.$$

This bound was proved in [37].

We now turn to the proof of Theorem 4.4.1. From now on, in the sequel, $\mathbb{M}$ will be a compact sub-Riemannian manifold of H-type such that for every smooth horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}^*} \geq \rho \|\eta\|^2_{\mathcal{H}^*},$$

with $\rho > 0$. Since $\mathbb{M}$ is of H-type, we have

$$\left\langle -\mathbf{J}^2(\eta), \eta \right\rangle_{\mathcal{H}^*} = \mathfrak{h} \|\eta\|^2_{\mathcal{H}^*},$$

and for every $Z \in \mathcal{V}$,

$$\mathbf{Tr}(J_Z^* J_Z) = d \|Z\|^2_{\mathcal{V}}.$$

From Theorem 4.3.1, we get therefore the lower bound

$$\lambda_1 \geq \frac{\rho d}{d - 1 + 3\mathfrak{h}}.$$

The key lemma in our rigidity result is the following result:

Lemma 4.4.2 *Let $f \in C^\infty(\mathbb{M})$ such that $Lf = -\lambda_1 f$ with $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$. Then f satisfies*

$$\nabla^2 f(X, Y) = -\frac{\lambda_1}{d} f \langle X, Y \rangle_{\mathcal{H}} - \frac{1}{2} T(X, Y) f, \quad \forall X, Y \in \mathcal{H}. \tag{4.9}$$

and

$$\nabla^2 f(X, Z) = \frac{2\lambda_1}{\rho_2} J_Z(X) f, \quad \forall X \in \mathcal{H}, Z \in \mathcal{V}. \tag{4.10}$$

Proof From (4.8) we have

$$\int_{\mathbb{M}} (Lf)^2 - 2\varepsilon \int_{\mathbb{M}} \langle d(Lf), df \rangle_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} + \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} \|df\|^2_{\mathcal{H}^*},$$

and thus, since $Lf = -\lambda_1 f$,

$$\lambda_1^2 \int_{\mathbb{M}} f^2 + 2\lambda_1 \varepsilon \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*} \geq \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} + \lambda_1 \left(\rho_1 - \frac{\kappa}{2\varepsilon}\right) \int_{\mathbb{M}} f^2. \tag{4.11}$$

On the other hand, from Lemma 4.3.2, we have

$$\begin{aligned} \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} \geq & \int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} + 2\varepsilon \int_{\mathbb{M}} \left\| \nabla_{\mathcal{H}} df - \frac{3}{2} \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df \right\|^2_{\mathcal{V}^*} \\ & + \frac{\rho_2}{2} \int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*} - \frac{5}{8\varepsilon} \int_{\mathbb{M}} \|df\|^2_{\mathcal{H}^*}. \end{aligned}$$

It is readily checked that

$$\|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\mathcal{H}^*} = \|\nabla^{2,\#}_{\mathcal{H}} f\|^2 + \frac{1}{4}\mathbf{Tr}(J^*_{\nabla_{\mathcal{V}} f} J_{\nabla_{\mathcal{V}} f}),$$

where $\nabla^{2,\#}_{\mathcal{H}} f$ denotes the symmetrization of the horizontal Hessian of f. Thus we have

$$\int_{\mathbb{M}} \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{2\varepsilon} \geq \int_{\mathbb{M}} \|\nabla^{2,\#}_{\mathcal{H}} f\|^2 + 2\varepsilon \int_{\mathbb{M}} \left\|\nabla_{\mathcal{H}} df - \frac{3}{2}\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\right\|^2_{\mathcal{V}^*}$$
$$+ \frac{3\rho_2}{4}\int_{\mathbb{M}} \|df\|^2_{\mathcal{V}^*} - \frac{5}{8\varepsilon}\lambda_1 \int_{\mathbb{M}} f^2.$$

Choosing ε such that $2\lambda_1\varepsilon = \frac{3\rho_2}{4}$ and using the last inequality in (4.11) gives eventually

$$\left(\lambda_1^2 - \lambda_1\left(\rho_1 - \frac{9\kappa}{8\varepsilon}\right)\right)\int_{\mathbb{M}} f^2 \geq \int_{\mathbb{M}} \|\nabla^{2,\#}_{\mathcal{H}} f\|^2 + 2\varepsilon\int_{\mathbb{M}} \left\|\nabla_{\mathcal{H}} df - \frac{3}{2}\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\right\|^2_{\mathcal{V}^*}.$$

From Cauchy-Schwarz inequality, given the value of λ_1, we always have

$$\left(\lambda_1^2 - \lambda_1\left(\rho_1 - \frac{9\kappa}{8\varepsilon}\right)\right)\int_{\mathbb{M}} f^2 \leq \int_{\mathbb{M}} \|\nabla^{2,\#}_{\mathcal{H}} f\|^2$$

This means that, necessarily

$$\left\|\nabla_{\mathcal{H}} df - \frac{3}{2}\mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\right\|_{\mathcal{V}^*} = 0,$$

and moreover that $\nabla^{2,\#}_{\mathcal{H}} f$ is a multiple of $g_{\mathcal{H}}$. This immediately implies (4.9) and (4.10). ■

We are now in position to prove Theorem 4.4.1.

Proof Let $f \in C^\infty(\mathbb{M})$ such that $Lf = -\lambda_1 f$ with $\lambda_1 = \frac{\rho d}{d-1+3\mathfrak{h}}$. From the previous lemma, we have

$$\nabla^2 f(X,Y) = -\frac{\lambda_1}{d} f\langle X,Y\rangle_{\mathcal{H}} - \frac{1}{2}T(X,Y)f, \quad \forall X,Y \in \mathcal{H}.$$

and

$$\nabla^2 f(X,Z) = \frac{2\lambda_1}{d} J_Z(X)f, \quad \forall X \in \mathcal{H}, Z \in \mathcal{V}.$$

The trick is now that, since $\mathbb{M}$ has transverse symmetries, $-L$ commutes with any $Z \in \mathcal{V}$ (see [10]), and thus Zf is also an eigenfunction for the same eigenvalue λ_1. In particular Zf also satisfies the equation (4.10). This gives for a horizontal vector field X and $Z \in \mathcal{V}$,

$$\nabla^3 f(X, Z, Z) = \frac{4\lambda_1^2}{d^2} J_Z^2(X) f.$$

From the H-type assumption, we deduce

$$\nabla^3 f(X, Z, Z) = -\frac{4\lambda_1^2}{d^2} \|Z\|_{\mathcal{V}}^2 X f.$$

Taking the trace and using the fact that both f and Zf are eigenfunctions of $-L$ with the same eigenvalue, we deduce that for any $Z \in \mathcal{V}$,

$$Z^2 f = -\frac{4\lambda_1^2}{d^2} \|Z\|_{\mathcal{V}}^2 f.$$

By polarization, it also implies that for every $Z, Z' \in \mathcal{V}$,

$$\frac{1}{2}(ZZ' + Z'Z) f = -\frac{4\lambda_1^2}{d^2} \langle Z, Z' \rangle_{\mathcal{V}} f. \tag{4.12}$$

Since $\mathbb{M}$ is compact, we easily see that $\mathcal{V}$ is a Lie algebra of compact type. We therefore can choose $g_{\mathcal{V}}$ to be a bi-invariant metric. We consider then the Riemannian metric on $\mathbb{M}$,

$$g_{2\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{2\varepsilon} g_{\mathcal{V}},$$

where $\varepsilon = \frac{2\lambda_1}{d}$. By denoting $\tilde{\nabla}$ the Levi-Civita connection associated to $g_{2\varepsilon}$, it is then an easy exercise to check that the previous relations imply then that for every smooth vector fields X, Y

$$\tilde{\nabla}^2 f(X, Y) = -\frac{\lambda_1}{d} f g_{2\varepsilon}(X, Y).$$

As a consequence of Obata's theorem [55], we deduce that $(\mathbb{M}, g_{2\varepsilon})$ is isometric to a sphere. Also by the very same Obata's theorem, the relations (4.12) imply that the Lie group $\mathbb{G}$ generated by $\mathcal{V}$ is a sphere itself. This implies that this group is either $\mathbf{U}(1)$ or $\mathbf{SU}(2)$. Morever, by the very definition of sub-Riemannian manifolds with transverse symmetries, $\mathbb{G}$ is seen to act properly on $\mathbb{M}$. We deduce that there is a Riemannian submersion with totally geodesic fibers

$$\mathbb{G} \to \mathbb{M} \to \mathbb{M}/\mathbb{G}.$$

The classification of Riemannian submersions with totally geodesic fibers of the sphere that was done in Escobales [26] completes our proof. ■

5. RIEMANNIAN FOLIATION

5.1 Introduction

In the work [10] the authors proved that on a sub-Riemannian manifold with transverse symmetries, assuming natural geometric conditions, the sub-Laplacian satisfies a generalized curvature dimension inequality. Among other things, this curvature dimension estimate implies Li-Yau inequalities for positive solutions of the heat equation [10], Gaussian lower and upper bounds for the subelliptic heat kernel [8, 10], log-Sobolev and isoperimetric inequalities [7, 12], volume and distance comparison estimates [9] and a Bonnet-Myers type theorem [10]. Recently, it has been pointed out by Elworthy [25] that sub-Riemannian manifolds with transverse symmetries can be seen as Riemannian manifolds with bundle like metrics which are foliated by totally geodesic leaves. The goal of the present work is two-fold:

- We actually prove that on any Riemannian foliation with bundle like metric and totally geodesic leaves, under natural geometric conditions, the horizontal Laplacian satisfies the curvature dimension estimate introduced in [10]. As a consequence, all the results proved in [7–12, 42] apply in this much more general case.

- We simplify the original approach of [10] by working out new Weitzenböck type identities for the horizontal Laplacian which we think are interesting in themselves. These Weitzenböck identities easily imply not only the curvature dimension estimate but also the stochastic completeness of the heat semigroup, which is a crucial ingredient to run the machinery developed in [10].

This chapter is organized as follows. In Section 2, we give the basic definitions and conventions that will be used throughout the text. In Section 3, we introduce a canonical one parameter family of horizontal Laplacians on one-forms and prove Weitzenböck-Bochner's type inequalities for this family of horizontal Laplacians. In Section 4, we prove the generalized curvature dimension inequality. We point out that, unlike many previous works on Riemannian foliations (see [65, 66] and the references therein), our results in Section 4 actually concern the sub-Riemannian geometry associated to the horizontal distribution and not only the basic geometry of the horizontal distribution.

5.2 Preliminaries

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $n+m$. We assume that $\mathbb{M}$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle like metric g and totally geodesic m-dimensional leaves (see the classical monograph by Tondeur [65] for the basic properties of such foliations) .

The sub-bundle $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of *vertical directions*. The sub-bundle $\mathcal{H}$ which is normal to $\mathcal{V}$ is referred to as the set of *horizontal directions*. The metric g can be split as

$$g = g_{\mathcal{H}} \oplus g_{\mathcal{V}},$$

and for later use, we introduce the one-parameter family of Riemannian metrics:

$$g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0,$$

which is going to play a pervasive role in the sequel. It is called the canonical variation of g, see Chapter 9 G in the monograph by Besse [17].

There is a canonical connection on $\mathbb{M}$, the Bott connection, which is given as follows:

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla_X^R Y), X, Y \in \Gamma^\infty(\mathcal{H}) \\ \pi_{\mathcal{H}}([X,Y]), X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}) \\ \pi_{\mathcal{V}}([X,Y]), X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}) \\ \pi_{\mathcal{V}}(\nabla_X^R Y), X, Y \in \Gamma^\infty(\mathcal{V}) \end{cases}$$

where ∇^R is the Levi-Civita connection and $\pi_{\mathcal{H}}$ (resp. $\pi_{\mathcal{V}}$) the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). It is easy to check that for every $\varepsilon > 0$, this connection satisfies $\nabla g_\varepsilon = 0$.

We define the horizontal gradient $\nabla_{\mathcal{H}} f$ of a function f as the projection of the Riemannian gradient of f on the horizontal bundle. Similarly, we define the vertical gradient $\nabla_{\mathcal{V}} f$ of a function f as the projection of the Riemannian gradient of f on the vertical bundle. The horizontal Laplacian is the generator of the symmetric Dirichlet form

$$\mathcal{E}_{\mathcal{H}}(f,g) = \int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle_{\mathcal{H}} d\mu.$$

It is a diffusion operator L on $\mathbb{M}$ which is symmetric on $C_0^\infty(\mathbb{M})$ with respect to the volume measure μ.

We now introduce some tensors that will play an important role in the sequel.

For $Z \in \Gamma^\infty(T\mathbb{M})$, there is a unique skew-symmetric endomorphism $J_Z : \mathcal{H}_x \to \mathcal{H}_x$ such that for all horizontal vector fields X and Y,

$$g_{\mathcal{H}}(J_Z(X), Y) = g_{\mathcal{V}}(Z, T(X,Y)). \tag{5.1}$$

where T is the torsion tensor of ∇. We then extend J_Z to be 0 on $\mathcal{V}_x$. If $Z_1, \cdots, Z_m$ is a local vertical frame, the operator $\sum_{\ell=1}^m J_{Z_\ell} J_{Z_\ell}$ does not depend on the choice of the frame and shall concisely be denoted by $\mathbf{J}^2$. For instance, if $\mathbb{M}$ is a K-contact manifold equipped with the Reeb foliation, then $\mathbf{J}$ is an almost complex structure, $\mathbf{J}^2 = -\mathbf{Id}_{\mathcal{H}}$.

The horizontal divergence of the torsion T is the $(1,1)$ tensor which is defined in a local horizontal frame $X_1, \cdots, X_n$ by

$$\delta_{\mathcal{H}} T(X) = \sum_{j=1}^{n} (\nabla_{X_j} T)(X_j, X).$$

The g-adjoint of $\delta_{\mathcal{H}}$ will be denoted $\delta_{\mathcal{H}}^* T$.

By declaring a one-form to be horizontal (resp. vertical) if it vanishes on the vertical bundle $\mathcal{V}$ (resp. on the horizontal bundle $\mathcal{H}$), the splitting of the tangent space

$$T_x \mathbb{M} = \mathcal{H}(x) \oplus \mathcal{V}(x)$$

gives a splitting of the cotangent space.

The metric g_ε induces then a metric on the cotangent bundle which we still denote g_ε. By using similar notations and conventions as before we have for every η in $T_x^* \mathbb{M}$,

$$\|\eta\|_\varepsilon^2 = \|\eta\|_{\mathcal{H}}^2 + \varepsilon \|\eta\|_{\mathcal{V}}^2.$$

By using the duality given by the metric g, $(1,1)$ tensors can also be seen as linear maps on the cotangent bundle $T^*\mathbb{M}$. More precisely, if A is a $(1,1)$ tensor, we will still denote by A the fiberwise linear map on the cotangent bundle which is defined as the g-adjoint of the dual map of A. The same convention will be made for any (r,s) tensor.

We define then the horizontal Ricci curvature $\mathfrak{Ric}_{\mathcal{H}}$ as the fiberwise symmetric linear map on one-forms such that for every smooth functions f, g,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg \rangle = \mathbf{Ricci}(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g),$$

where **Ricci** is the Ricci curvature of the connection ∇.

If V is a horizontal vector field and $\varepsilon > 0$, we consider the fiberwise linear map from the space of one-forms into itself which is given for $\eta \in \Gamma^\infty(T^*\mathbb{M})$ and $Y \in \Gamma^\infty(T\mathbb{M})$ by

$$\mathfrak{T}_V^\varepsilon \eta(Y) = \begin{cases} \frac{1}{\varepsilon}\eta(J_Y V), \quad Y \in \Gamma^\infty(\mathcal{V}) \\ -\eta(T(V,Y)), Y \in \Gamma^\infty(\mathcal{H}) \end{cases}$$

We observe that $\mathfrak{T}_V^\varepsilon$ is skew-symmetric for the metric g_ε so that $\nabla - \mathfrak{T}^\varepsilon$ is a g_ε-metric connection.

If $Z_1, \cdots, Z_m$ is a local vertical frame of the leaves, we denote

$$\mathfrak{J}(\eta) = \sum_{\ell=1}^m J_{Z_\ell}(\iota_{Z_\ell} d\eta_\mathcal{V}),$$

where $\eta_\mathcal{V}$ is the the projection of η to the vertical cotangent bundle. It does not depend on the choice of the frame,

If η is a one-form, we define the horizontal gradient in a local adapted frame of η as the $(0,2)$ tensor

$$\nabla_\mathcal{H}\eta = \sum_{i=1}^n \nabla_{X_i}\eta \otimes \theta_i.$$

We denote by $\nabla_\mathcal{H}^\# \eta$ the symmetrization of $\nabla_\mathcal{H}\eta$.

Similarly, we will use the notation

$$\mathfrak{T}_\mathcal{H}^\varepsilon \eta = \sum_{i=1}^n \mathfrak{T}_{X_i}^\varepsilon \eta \otimes \theta_i.$$

Finally, we will still denote by L the covariant extension on one-forms of the horizontal Laplacian. In a local horizontal frame, we have thus

$$L = \sum_{i=1}^d \nabla_{X_i}\nabla_{X_i} - \nabla_{\nabla_{X_i}X_i}.$$

5.3 Bochner-Weitzenböck formulas for the horizontal Laplacian

For $\varepsilon > 0$, we consider the following operator which is defined on one-forms by

$$\square_\varepsilon = -(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H})^*(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H}) - \frac{1}{\varepsilon}\mathbf{J}^2 + \frac{1}{\varepsilon}\delta_\mathcal{H} T - \mathfrak{Ric}_\mathcal{H},$$

where the adjoint is understood with respect to the metric g_ε. It is easily seen that, in a local horizontal frame,

$$-(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H})^*(\nabla_\mathcal{H} - \mathfrak{T}^\varepsilon_\mathcal{H}) = \sum_{i=1}^n (\nabla_{X_i} - \mathfrak{T}^\varepsilon_{X_i})^2 - (\nabla_{\nabla_{X_i} X_i} - \mathfrak{T}^\varepsilon_{\nabla_{X_i} X_i}), \tag{5.2}$$

The following theorem is the main result of the section

Theorem 5.3.1 *For every $f \in C^\infty(\mathbb{M})$, we have*

$$dLf = \square_\varepsilon df,$$

and for every $\eta \in \Gamma^\infty(T^*\mathbb{M})$,

$$\begin{aligned}
&\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta\rangle_\varepsilon \\
=&\|\nabla_\mathcal{H}\eta - \mathfrak{T}^\varepsilon_\mathcal{H}\eta\|_\varepsilon^2 + \langle \mathfrak{Ric}_\mathcal{H}(\eta), \eta\rangle_\mathcal{H} - \langle \delta_\mathcal{H} T(\eta), \eta\rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_\mathcal{H} \\
\geq& \frac{1}{n}\left(\mathbf{Tr}_\mathcal{H} \nabla^\#_\mathcal{H}\eta\right)^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(J_\eta^2) + \langle \mathfrak{Ric}_\mathcal{H}(\eta), \eta\rangle_\mathcal{H} - \langle \delta_\mathcal{H} T(\eta), \eta\rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_\mathcal{H}
\end{aligned}$$

The remainder of the section is devoted to the proof of this result.

We proceed in several steps and divide the proof into several lemmas. Since the statement is local, we can assume that the Riemannian foliation comes from a Riemannian submersion with totally geodesic fibers. We fix $x \in \mathbb{M}$ throughout the proof.

Let $X_1, \cdots, X_n$ be a local orthonormal horizontal frame around x consisting of basic vector fields for the submersion. We can assume that, at x, $\nabla_{X_i} X_j = 0$. Let now $Z_1, \cdots, Z_m$ be a local orthonormal vertical frame around x. Since X_i is basic, the vector field $[X_i, Z_m]$ is tangent to the leaves. We write the structure constants in that local frame:

$$[X_i, X_j] = \sum_{k=1}^{n} \omega_{ij}^k X_k + \sum_{k=1}^{m} \gamma_{ij}^k Z_k$$

$$[X_i, Z_k] = \sum_{j=1}^{m} \beta_{ik}^j Z_j,$$

and observe that at the center x of the frame, we have $\omega_{ij}^k = 0$. Moreover, since X_i is basic and the submersion has totally geodesic fibers, the flow generated by X_i induces an isometry between the leaves (see Besse [17], Chapter 9), as a consequence we have the skew-symmetry,

$$\beta_{ik}^j = -\beta_{ij}^k.$$

It is easy to see that we can also assume that, at the center x, $\beta_{ij}^k = 0$ (see for instance [35], Corollary 2.22).

The dual coframe of $\{X_1, \cdots, X_n, Z_1, \cdots, Z_m\}$ will be denoted $\{\theta_1, \cdots, \theta_n, \nu_1, \cdots, \nu_m\}$ and a generic one-form η will be written, $\eta = \sum_{i=1}^n f_i \theta_i + \sum_{\ell=1}^m g_\ell \nu_\ell$.

Lemma 5.3.2 *At x,*

- $\mathbf{Ricci}(X_i, X_k) = \sum_{j=1}^n \left(\frac{1}{2} X_j (\omega_{ik}^j - \omega_{ij}^k - \omega_{kj}^i) - X_i \omega_{jk}^j \right)$.
- $\mathbf{Ricci}(Z_\ell, X_k) = -\sum_{j=1}^n Z_\ell \omega_{jk}^j = 0$.
- $\mathfrak{J}(\eta) = \sum_{i,j=1}^n \sum_{\ell=1}^m \gamma_{ji}^\ell (X_i g_\ell) \theta_j$.
- $\delta_{\mathcal{H}} T(\eta) = -\sum_{i,j=1}^n \sum_{\ell=1}^m \left(X_i \gamma_{ij}^\ell \right) f_j \nu_\ell$.
- $\delta_{\mathcal{H}}^* T(\eta) = \sum_{i,j=1}^n \sum_{k=1}^m \left(X_j \gamma_{ij}^k \right) g_k \theta_i$.
- $\mathfrak{T}_{X_i}^\varepsilon \eta = \sum_{j=1}^n \sum_{\ell=1}^m \left(\gamma_{ij}^\ell g_\ell \theta_j - \frac{1}{\epsilon} \gamma_{ij}^\ell f_j \nu_\ell \right)$

Proof The computations are routine. We just point out that the vanishing of $\mathbf{Ricci}(Z_\ell, X_k)$ comes from the fact that since X_k and $\nabla_{X_i} X_k$ basic and $[X_i, Z_\ell]$ is tangent to the leaves, we have at x, $\nabla_{Z_\ell} X_k = \nabla_{Z_\ell} \nabla_{X_i} X_k = \nabla_{[X_i, Z_\ell]} X_k = 0$.

■

Lemma 5.3.3 *Let $\square_\infty$ be the operator defined on one-forms by*

$$\square_\infty = L + 2\mathfrak{J} - \mathfrak{Ric}_{\mathcal{H}} + \delta^*_{\mathcal{H}} T,$$

then for any $f \in C^\infty(\mathbb{M})$,

$$dLf = \square_\infty df.$$

Proof We compute,

$$\begin{aligned} dLf - Ldf = & \sum_{i=1}^n ([X_i, L]f)\theta_i - (X_i f)L\theta_i - \sum_{j=1}^n 2(X_j X_i f)\nabla_{X_j}\theta_i \\ & + \sum_{\ell=1}^m ([Z_\ell, L]f)\nu_\ell - (Z_\ell f)L\nu_\ell - \sum_{j=1}^n 2(X_j Z_\ell f)\nabla_{X_j}\nu_\ell. \end{aligned} \tag{5.3}$$

Now, at the center x of the frame, we have $\nabla_{X_j}\theta_i = \nabla_{X_j}\nu_\ell = 0$, and

$$L\theta_i = \sum_{j,k=1}^n (-X_j \Gamma^i_{jk})\theta_k, \quad L\nu_\ell = \sum_{j=1}^n \sum_{k=1}^m (-X_j \beta^\ell_{jk})\nu_k. \tag{5.4}$$

where the Γ^k_{ij}'s are the Cristofell symbols of the Bott's connection. We also easily compute

$$[Z_\ell, L]f = \sum_{j=1}^m (-\sum_{i=1}^n X_i \beta^j_{i\ell}) Z_j f$$

and

$$[X_i, L]f = \sum_{j=1}^n \sum_{l=\ell}^m 2\gamma^\ell_{ij} X_j Z_\ell f + \sum_{\ell=1}^m (\sum_{j=1}^n X_j \gamma^\ell_{ij}) Z_\ell f + \sum_{j,k=1}^n (X_k \omega^j_{ik} - X_i \omega^k_{jk}) X_j f. \tag{5.5}$$

If we plug this in (5.3), then the second line of (5.3) turns out to be 0 and we deduce from Lemma 5.3.2 that at x, we have

$$dLf - Ldf = 2\mathfrak{J}(df) - \mathfrak{Ric}_{\mathcal{H}}(df) + \delta^*_{\mathcal{H}} T(df).$$

This completes the proof. ■

Let us consider the map $\mathcal{T}\colon \Gamma^\infty(\wedge^2 T^*\mathbb{M}) \to \Gamma^\infty(T^*\mathbb{M})$ which is given in the local frame by,

$$\mathcal{T}(\theta_i \wedge \theta_j) = -\gamma_{ij}^\ell \nu_\ell, \quad \mathcal{T}(\theta_i \wedge \nu_k) = \mathcal{T}(\nu_k \wedge \nu_\ell) = 0.$$

Lemma 5.3.4 *For $\varepsilon > 0$*

$$\square_\varepsilon = \square_\infty - \frac{2}{\varepsilon}\mathcal{T} \circ d.$$

Proof A direct computation shows that, at the center x, for any $\eta = \sum_{j=1}^n f_j\theta_j + \sum_{k=1}^m g_k\nu_k$,

$$\begin{aligned}
&-(\nabla_\mathcal{H} - \mathfrak{T}_\mathcal{H}^\varepsilon)^*(\nabla_\mathcal{H} - \mathfrak{T}_\mathcal{H}^\varepsilon)\eta \\
=& \sum_{i,j=1}^n \left(X_i^2 f_j \theta_j - \sum_{\ell=1}^m X_i(\gamma_{ij}^\ell g_\ell) \right) \theta_j \\
&+ \sum_{\ell=1}^m \left(\sum_{i=1}^n X_i^2 g_\ell + \frac{1}{\varepsilon} \sum_{i,j=1}^n X_i(\gamma_{ij} f_j) + \sum_{i=1}^n \sum_{k=1}^m X_i(\beta_{ik}^\ell g_k) \right) \nu_\ell \\
&+ \frac{1}{\varepsilon} \sum_{i,j=1}^n \sum_{k=1}^m \left(X_i f_j - \sum_{\ell=1}^m \gamma_{ij}^\ell g_\ell \right) \gamma_{ij}^k \nu_k \\
&- \sum_{i,h=1}^n \sum_{\ell=1}^m \left(X_i g_\ell + \frac{1}{\varepsilon} \sum_{j=1}^n \gamma_{ij}^\ell f_j + \sum_{k=1}^m \beta_{ik}^\ell g_k \right) \gamma_{ih}^\ell \theta_h
\end{aligned}$$

Therefore, we have

$$\left(-(\nabla_\mathcal{H} - \mathfrak{T}_\mathcal{H}^\varepsilon)^*(\nabla_\mathcal{H} - \mathfrak{T}_\mathcal{H}^\varepsilon) - \frac{1}{\varepsilon}\mathbf{J}^2 \right)\eta = \left(L + 2\mathfrak{J} - \frac{2}{\varepsilon}\mathcal{T} \circ d + \delta_\mathcal{H}^* T - \frac{1}{\varepsilon}\delta_\mathcal{H} T \right)\eta.$$

By using the definition of $\square_\infty$ we immediately obtain the conclusion. ■

Lemma 5.3.5 *For any $\eta \in \Gamma^\infty(T^*\mathbb{M})$,*

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta \rangle_\varepsilon = \|\nabla_\mathcal{H}\eta - \mathfrak{T}_\mathcal{H}^\varepsilon \eta\|_\varepsilon^2 + \langle \mathfrak{Ric}_\mathcal{H}(\eta), \eta \rangle_\mathcal{H} - \langle \delta_\mathcal{H} T(\eta), \eta \rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta \rangle_\mathcal{H}.$$

Proof First note that

$$\square_\varepsilon = \square_\infty - \frac{2}{\varepsilon}\mathcal{T} \circ d = L + 2\mathfrak{J} - \mathfrak{Ric}_\mathcal{H} + \delta_\mathcal{H}^* T - \frac{2}{\varepsilon}\mathcal{T} \circ d,$$

therefore it is equivalent to show that

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \left\langle \left(L + 2\mathfrak{J} - \frac{2}{\varepsilon}\mathcal{T}\circ d \right)\eta, \eta \right\rangle_\varepsilon = \|\nabla_\mathcal{H}\eta - \mathfrak{T}_\mathcal{H}^\varepsilon\eta\|_\varepsilon^2 + \frac{1}{\varepsilon}\left\langle \mathbf{J}^2(\eta), \eta \right\rangle_\mathcal{H}. \quad (5.6)$$

We now compute that, at the center x,

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \left\langle \left(L + 2\mathfrak{J} - \frac{2}{\varepsilon}\mathcal{T}\circ d \right)\eta, \eta \right\rangle_\varepsilon \quad (5.7)$$

$$= \|\nabla_\mathcal{H}\eta\|_\mathcal{H}^2 + \varepsilon\|\nabla_\mathcal{V}\eta\|_\mathcal{V}^2 + 2\sum_{i,j=1}^n\sum_{k,\ell=1}^m \gamma_{ij}^\ell (X_i g_\ell) f_j - 2\sum_{i,j=1}^n\sum_{k,\ell=1}^m \gamma_{ij}^\ell (X_i f_j - \frac{1}{2}\gamma_{ij}^k g_k) g_\ell.$$

and

$$\|\nabla_\mathcal{H}\eta - \mathfrak{T}_\mathcal{H}^\varepsilon\eta\|_\varepsilon^2 = \sum_{i,j=1}^n \left(X_i f_j - \sum_{\ell=1}^m \gamma_{ij}^\ell g_\ell \right)^2 + \varepsilon\sum_{\ell=1}^m\sum_{i=1}^n \left(X_i g_\ell + \frac{1}{\epsilon}\sum_{j=1}^n \gamma_{ij}^\ell f_j \right)^2$$

$$= \|\nabla_\mathcal{H}\eta\|_\mathcal{H}^2 + \varepsilon\|\nabla_\mathcal{V}\eta\|_\mathcal{V}^2 - 2\sum_{i,j=1}^n\sum_{\ell=1}^m (X_i f_j)\gamma_{ij}^\ell g_\ell + \sum_{i,j=1}^n\sum_{k,\ell=1}^m \gamma_{ij}^\ell\gamma_{ij}^k g_\ell g_k$$

$$+ 2\varepsilon\sum_{i=1}^n\sum_{\ell=1}^m (X_i g_\ell)\left(\frac{1}{\varepsilon}\sum_{j=1}^n \gamma_{ij}^\ell f_j \right) + \varepsilon\sum_{\ell=1}^m\sum_{i=1}^n \left(\frac{1}{\epsilon}\sum_{j=1}^n \gamma_{ij}^\ell f_j \right)^2$$

The claim easily follows. ■

Proposition 5.3.1 *For every $\eta \in \Gamma^\infty(T^*\mathbb{M})$,*

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \Box_\varepsilon\eta, \eta\rangle_\varepsilon$$
$$\geq \frac{1}{n}\left(\mathbf{Tr}_\mathcal{H}\nabla_\mathcal{H}^\#\eta \right)^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(J_\eta^2) + \langle \mathfrak{Ric}_\mathcal{H}(\eta), \eta\rangle_\mathcal{H} - \langle \delta_\mathcal{H}T(\eta), \eta\rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_\mathcal{H}$$

Proof Due to Lemma 5.3.5 it amounts to prove that

$$\|\nabla_\mathcal{H}\eta - \mathfrak{T}_\mathcal{H}^\varepsilon\eta\|_\varepsilon^2 \geq \frac{1}{n}\left(\mathbf{Tr}_\mathcal{H}\nabla_\mathcal{H}^\#\eta \right)^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(J_\eta^2).$$

For any $\eta \in \Gamma^\infty(T^*\mathbb{M})$, we have at x

$$\nabla_\mathcal{H}^\#(\eta) = \frac{1}{2}\sum_{i,j=1}^n ((X_i f_j)\theta_j + (X_j f_i)\theta_i) + \sum_{\ell=1}^m (X_i g_\ell)\,\nu_\ell.$$

Easy computations show that

$$\|\nabla_\mathcal{H}\eta - \mathfrak{T}_\mathcal{H}^\varepsilon\eta\|_\mathcal{H}^2 = \sum_{i,j=1}^n \left(X_i f_j - \sum_{\ell=1}^m \gamma_{ij}^\ell g_\ell \right)^2 = \|\nabla_\mathcal{H}^\#\eta\|_\mathcal{H}^2 + \frac{1}{4}\sum_{i,j=1}^n \left(\sum_{\ell=1}^m \gamma_{ij}^\ell g_\ell \right)^2,$$

therefore, from Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\|\nabla_{\mathcal{H}}\eta - \mathfrak{T}^{\varepsilon}_{\mathcal{H}}\eta\|^2_{\varepsilon} &= \|\nabla^{\#}_{\mathcal{H}}\eta\|^2_{\varepsilon} + \frac{1}{4}\sum_{i,j=1}^{n}\left(\sum_{\ell=1}^{m}\gamma^{\ell}_{ij}g_{\ell}\right)^2 + \varepsilon\|\nabla_{\mathcal{H}}\eta - \mathfrak{T}^{\varepsilon}_{\mathcal{H}}\eta\|^2_{V} \\ &\geq \frac{1}{n}\left(\mathbf{Tr}_{\mathcal{H}}\nabla^{\#}_{\mathcal{H}}\eta\right)^2 - \frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J^2_{\eta}).\end{aligned}$$

■

5.4 Stochastic completeness

Throughout the section we consider, as above, a smooth connected manifold $\mathbb{M}$ which is equipped with a Riemannian foliation with bundle like metric g and totally geodesic leaves. We moreover assume that the metric g is complete and that the horizontal distribution $\mathcal{H}$ of the foliation is bracket-generating and Yang-Mills (see Besse [17], Definition 9.35). The hypothesis that $\mathcal{H}$ is bracket generating implies that the horizontal Laplacian L is subelliptic and it is easily seen that with our notations the Yang-Mills condition is equivalent to the fact that

$$\delta_{\mathcal{H}}T = 0.$$

The operator

$$\square_{\varepsilon} = -(\nabla_{\mathcal{H}} - \mathfrak{T}^{\varepsilon}_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^{\varepsilon}_{\mathcal{H}}) - \frac{1}{\varepsilon}\mathbf{J}^2 - \mathfrak{Ric}_{\mathcal{H}}.$$

that we introduced in the previous section is then symmetric for the metric g_{ε}.

In this section, we also assume that for every horizontal one-form $\eta \in \Gamma^{\infty}(\mathcal{H}^*)$,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta\rangle_{\mathcal{H}} \geq -K\|\eta\|^2_{\mathcal{H}}, \quad -\langle \mathbf{J}^2\eta, \eta\rangle_{\mathcal{H}} \leq \kappa\|\eta\|^2_{\mathcal{H}},$$

with $K, \kappa \geq 0$.

The completeness of the metric g implies that the horizontal Laplacian L is essentially self-adjoint on the space of smooth and compactly supported functions and that the operator $\square_\varepsilon$ is essentially self-adjoint on the space of smooth and compactly supported one-forms (see the argument in Lemma 4.3 of [6]).

Since $\square_\varepsilon$ is essentially self-adjoint, it admits a unique self-adjoint extension which generates thanks to the spectral theorem a semigroup $Q_t^\varepsilon = e^{t\square_\varepsilon}$. We will denote by $P_t = e^{tL}$ the semigroup generated by L. We have the following commutation property:

Lemma 5.4.1 *If $f \in C_0^\infty(\mathbb{M})$, then for every $t \geq 0$,*

$$dP_t f = Q_t^\varepsilon df.$$

Proof Let $\eta_t = Q_t^\varepsilon df$. By essential self-adjointness, it is the unique solution in L^2 of the heat equation

$$\frac{\partial \eta}{\partial t} = \square_\varepsilon \eta,$$

with initial condition $\eta_0 = df$. From the fact that

$$dL = \square_\varepsilon d,$$

we see that $\alpha_t = dP_t f$ solves the heat equation

$$\frac{\partial \alpha}{\partial t} = \square_\varepsilon \alpha$$

with the same initial condition $\alpha_0 = df$. In order to conclude, we thus just need to prove that for every $t \geq 0$, $dP_t f$ is in L^2 . Let us denote by L^V the vertical (leaf) Laplacian. The Laplace-Beltrami operator of $\mathbb{M}$ is therefore $\Delta = L + L^V$. Since the leaves are totally geodesic, Δ commutes with L on C^2 functions (see [16]). Moreover from the spectral theorem, $Le^{t\Delta}$ maps $C_0^\infty(\mathbb{M})$ into $L^2(\mathbb{M})$. We deduce by essential self-adjointness that $Le^{t\Delta} = e^{t\Delta}L$. Similarly we obtain $e^{sL}e^{t\Delta} = e^{t\Delta}e^{sL}$ which implies $\Delta e^{sL} = e^{sL}\Delta$. As a consequence we have that for every $t \geq 0$, $dP_t f$ is in L^2. ■

Theorem 5.4.2 *For every $\varepsilon > 0$, $t \geq 0, x \in \mathbb{M}$ and $f \in C_0^\infty(\mathbb{M})$,*

$$\|dP_t f(x)\|_\varepsilon \leq e^{\left(K + \frac{\kappa}{\varepsilon}\right)t} P_t \|df\|_\varepsilon(x).$$

As a consequence, the heat semigroup is conservative that is for every $t \geq 0$, $P_t 1 = 1$.

Proof The idea is to use the Feynman-Kac stochastic representation of Q_t^ε. We denote by $(X_t)_{t\geq 0}$ the symmetric diffusion process generated by $\frac{1}{2}L$ and denote by $\mathbf{e}$ its lifetime. Consider the process $\tau_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_{X_0}\mathbb{M}$ which is the solution of the following covariant Stratonovitch stochastic differential equation:

$$d\left[\tau_t^\varepsilon \alpha(X_t)\right] = \tau_t^\varepsilon \left(\nabla_{\circ dX_t} - \mathfrak{T}^\varepsilon_{\circ dX_t} - \frac{1}{2}\left(\frac{1}{\varepsilon}\mathbf{J}^2 + \mathfrak{Ric}_{\mathcal{H}}\right) dt \right) \alpha(X_t), \quad \tau_0^\varepsilon = \mathbf{Id}, \tag{5.8}$$

where α is any smooth one-form. By using Gronwall's lemma, we have for every $t \geq 0$,

$$\|\tau_t^\varepsilon \alpha(X_t)\|_\varepsilon \leq e^{\frac{1}{2}\left(K+\frac{\kappa}{\varepsilon}\right)t} \|\alpha(X_0)\|_\varepsilon.$$

By the Feynman-Kac formula, we have for every smooth and compactly supported one-form

$$Q_{t/2}\eta(x) = \mathbb{E}_x\left(\tau_t \eta(X_t) \mathbf{1}_{t<\mathbf{e}}\right).$$

Since $dP_t = Q_t^\varepsilon d$, it follows easily that

$$\|dP_t f(x)\|_\varepsilon \leq e^{\left(K+\frac{\kappa}{\varepsilon}\right)t} P_t \|df\|_\varepsilon(x).$$

It is well-known that this type of gradient bound implies the stochastic completeness of P_t. ∎

5.5 Curvature-dimension inequality, Li-Yau estimates and Bonnet-Myers type theorem

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $n+m$. We assume that $\mathbb{M}$ is equipped with a Riemannian foliation $\mathcal{F}$ with bundle like metric g and totally geodesic m-dimensional leaves for which the horizontal distribution is Yang-Mills. We also assume that $\mathbb{M}$ is complete and that globally on $\mathbb{M}$, for every $\eta_1 \in \Gamma^\infty(\mathcal{H}^*)$ and $\eta_2 \in \Gamma^\infty(\mathcal{V}^*)$,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta_1), \eta_1\rangle_{\mathcal{H}} \geq \rho_1 \|\eta_1\|^2_{\mathcal{H}}, \quad -\langle \mathbf{J}^2 \eta_1, \eta_1\rangle_{\mathcal{H}} \leq \kappa \|\eta_1\|^2_{\mathcal{H}}, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J^2_{\eta_2}) \geq \rho_2 \|\eta_2\|^2_{\mathcal{V}},$$

for some $\rho_1 \in \mathbb{R}$, $\kappa, \rho_2 > 0$. The third assumption can be thought as a uniform bracket generating condition of the horizontal distribution $\mathcal{H}$ and from Hörmander's theorem, it implies that the horizontal Laplacian L is a subelliptic diffusion operator. We insist that for the following results below to be true, the positivity of ρ_2 is required.

We introduce the following operators defined for $f, g \in C^\infty(\mathbb{M})$,

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - gLf - fLg) = \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle_{\mathcal{H}}$$

$$\Gamma^{\mathcal{V}}(f,g) = \langle \nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} g \rangle_{\mathcal{V}}$$

and their iterations which are defined by

$$\Gamma_2(f,g) = \frac{1}{2}(L(\Gamma(f,g)) - \Gamma(g, Lf) - \Gamma(f, Lg))$$

$$\Gamma_2^{\mathcal{V}}(f,g) = \frac{1}{2}(L(\Gamma^{\mathcal{V}}(f,g)) - \Gamma^{\mathcal{V}}(g, Lf) - \Gamma^{\mathcal{V}}(f, Lg))$$

As a consequence of Theorem 5.3.1, we obtain the curvature dimension inequality introduced in [10].

Theorem 5.5.1 *For every $f, g \in C^\infty(\mathbb{M})$, and $\varepsilon > 0$,*

$$\Gamma_2(f,f) + \varepsilon \Gamma_2^{\mathcal{V}}(f,f) \geq \frac{1}{n}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\varepsilon}\right)\Gamma(f,f) + \rho_2 \Gamma^{\mathcal{V}}(f,f),$$

and

$$\Gamma(f, \Gamma^{\mathcal{V}}(f)) = \Gamma^{\mathcal{V}}(f, \Gamma(f)).$$

Proof From Theorem 3.1, we have for every $\eta \in \Gamma^\infty(T^*\mathbb{M})$,

$$\frac{1}{2}L\|\eta\|_\varepsilon^2 - \langle \Box_\varepsilon \eta, \eta \rangle_\varepsilon \geq \frac{1}{n}\left(\mathbf{Tr}_{\mathcal{H}} \nabla_{\mathcal{H}}^{\#} \eta\right)^2 - \frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_\eta^2) + \langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}}.$$

Using this inequality with $\eta = df$ and taking into account the assumptions

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta_1), \eta_1 \rangle_{\mathcal{H}} \geq \rho_1 \|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2 \eta_1, \eta_1 \rangle_{\mathcal{H}} \leq \kappa \|\eta_1\|_{\mathcal{H}}^2, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_{\eta_2}^2) \geq \rho_2 \|\eta_2\|_{\mathcal{V}}^2,$$

immediately yields the expected result. The intertwining $\Gamma(f, \Gamma^{\mathcal{V}}(f)) = \Gamma^{\mathcal{V}}(f, \Gamma(f))$ is proved in Appendix A in [25] and easy to check in a local frame. ■

Combining Theorems 5.4.2 and 5.5.1, we see then that all the results proved in the works [7–12, 42] apply. We obtain therefore, among many other things, the following results which are completely new in the context of Riemannian foliations:

1) **Li-Yau type inequalities:** ([10]) For any bounded $f \in C^\infty(\mathbb{M})$, such that $f, \sqrt{\Gamma(f)}, \sqrt{\Gamma^{\mathcal{V}}(f)} \in L^2_\mu(\mathbb{M})$, $f \geq 0$, $f \neq 0$, the following inequality holds for $t > 0$:

$$\begin{aligned}
&\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^{\mathcal{V}}(\ln P_t f) \\
&\leq \left(1 + \frac{3\kappa}{2\rho_2} - \frac{2\rho_1}{3} t\right) \frac{L P_t f}{P_t f} + \frac{n\rho_1^2}{6} t - \frac{n\rho_1}{2}\left(1 + \frac{3\kappa}{2\rho_2}\right) + \frac{n\left(1+\frac{3\kappa}{2\rho_2}\right)^2}{2t}.
\end{aligned}$$

2) **Gaussian lower and upper bounds for the horizontal heat kernel:** ([8]) If $\rho_1 \geq 0$, then for any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(n, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \to 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has
$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{D d(x,y)^2}{n(4-\varepsilon)t}\right) \leq p(x,y,t) \leq \frac{C(\varepsilon)}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right).$$
Here $D = \left(1 + \frac{3\kappa}{2\rho_2}\right) n$ and $d(x,y)$ is the sub-Riemannian distance between x and y.

3) **Bonnet-Myers theorem:** ([10]) Suppose that $\rho_1 > 0$. Then, the manifold $\mathbb{M}$ is compact and the sub-Riemannian diameter of $\mathbb{M}$ satisfies the bound
$$\operatorname{diam} \mathbb{M} \leq 2\sqrt{3}\pi \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2}\left(1 + \frac{3\kappa}{2\rho_2}\right) n}. \tag{5.9}$$

We mention that in the Bonnet-Myers theorem, the bound
$$\operatorname{diam} \mathbb{M} \leq 2\sqrt{3}\pi \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2}\left(1 + \frac{3\kappa}{2\rho_2}\right) n}.$$
is not sharp, as can be checked in some examples like the Hopf fibrations. This is because the method we use in [10] is an adaption of the energy-entropy inequality methods developped by Bakry in [3]. Even in the Riemannian case, Bakry methods are known to lead to non sharp constants.

Finally, at last, we observe that the methods of [13] can be adapted to the present framework and that the following result can be proved:

Proposition 5.5.1 *Assume* $\rho_1 > 0$. *Then the first eigenvalue* λ_1 *of the horizontal Laplacian* $-L$ *satisfies*

$$\lambda_1 \geq \frac{\rho_1}{1 - \frac{1}{n} + \frac{3\kappa}{4\rho_2}}.$$

As a consequence of the Obata theorem proved in [13], this bound is sharp.

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